Renormalization Group and Resonances in Non-relativistic QED

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Earlier work: Bach-Fröhlich-IMS, Bach-Fröhlich-Pizzo,
Fröhlich-Griesemer-IMS, Abou Salem-Faupin-Fröhlich-IMS

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Problem and Results

Our goal is to prove existence of the ground states and resonances in the non-relativistic QED. Namely, for a quantum-mechanical system of particles coupled to quantized electromagnetic field we show that

- The *ground state* of the particle system is *stable* as the coupling is turned on, while
- The excited states, generically, are not. They turn into *resonances*.

The resonances are responsible for processes of emission and absorption of the electro-magnetic radiation.

Earlier Resuilts

The existence (and uniqueness) of the ground state:
Griesemer-Lieb-Loss, Lieb-Loss, Bach-Fröhlich- Sigal, Hiroshima,
Hiroshima-Spohn, Arai-Hirokawa, Bach-Fröhlich-Pizzo
The existence of the resonances, radiative correctoions and
life-times: Bach-Fröhlich-Sigal (for confined potentials)
Related results:
The asymptotic stability of the ground state (local decay):
Fröhlich- Griesemer-Sigal,
Bach-Fröhlich-Sigal,Bach-Fröhlich-Sigal-Soffer.
Scattering amplitudes and asymptotic expansions:
Bach-Fröhlich-Pizzo.
The survival probabilities of excited states: Abou
Salem-Faupin-Fröhlich-Sigal and Hasler-Herbst-Huber,
Bach-Fröhlich-Sigal, Mück, King.
Atoms with dynamic nuclei: Faupin, Amour- Grébert - Guillot.

Quantized Electromagnetic Field

The quantized electromagnetic field is described by the *quantized vector potential*

$$A(y) = \int (e^{iky} a(k) + e^{-iky} a^*(k)) \chi(k) \frac{d^3 k}{\sqrt{|k|}},$$

in the Coulomb gauge ($\mathrm{div} A(y) = 0$). Here χ is an *ultraviolet cut-off*:
$\chi(k) = 1$ in a neighborhood of $k = 0$ and say $|\chi(k)| \lesssim \langle k/\kappa \rangle^{-3}$.

The dynamics of quantized EM field is given by the Hamiltonian

$$H_f \ = \ \int d^3 k \, \omega(k) a^*(k) \cdot a(k),$$

where $\omega(k) \ = \ |k|$ is the dispersion law.

Ultra-violet Cut-off

Let the ultra-violet cut-off $\chi(k)$ decay on the scale κ. We assume that the cut-off energy,

$$\hbar c\kappa \gg \alpha^2 mc^2, \text{ ionization energy },$$

a *characteristic energy of the particle motion*, or $\alpha^2 \ll \kappa$ in our units. On the other hand, we assume

$$\hbar c\kappa \ll mc^2, \text{ the rest energy of the the electron,}$$

where the relativistic effects, such as electron-positron pair creation, vacuum polarization and relativistic recoil, take place. Combining the last two conditions we arrive at

$$\alpha^2 \ll \kappa \ll 1 \ \ (\alpha^2 mc/\hbar \ll \kappa \ll mc/\hbar).$$

After the rescaling $x \to \alpha^{-1}x$ and $k \to \alpha^2 k$ the new cut-off momentum scale, $\kappa' = \alpha^{-2}\kappa$, satisfies

$$1 \ll \kappa' \ll \alpha^{-2},$$

which is easily accommodated by our estimates.

Hamiltonian

The mathematical framework of the non-relativistic QED is given in terms of the time-dependent Schrödinger equation with the standard quantum Hamiltonian (after rescaling)

$$H_g^{SM} = \sum_{j=1}^{n} \frac{1}{2m_j}(i\nabla_{x_j} + gA(x_j))^2 + V(x) + H_f,$$

acting on the Hilbert space $\mathcal{H} = \mathcal{H}_p \otimes \mathcal{H}_f$. Here $x = (x_1, \ldots, x_n)$, $V(x)$ is the total potential affecting particles and $g > 0$ is a coupling constant related to the particle charge.

Besides $V(x)$, H_g^{SM} depends on *two free parameters*:
The coupling constant g ($\sim$ *electron charge*) and
the ultraviolet cut-off κ ($\sim$ *electron renormalized mass*).

Complex Deformations

Quantum resonances manifest themselves in three different ways:

- Eigenvalues of complexly deformed Hamiltonian;
- Poles of the meromorphic continuation of the resolvent across the continuous spectrum;
- Metastable states.

To define the resonances for the Hamiltonian H_g^{SM} we pass to the one-parameter (deformation) family

$$H_{g\theta}^{SM} := U_\theta H_g^{SM} U_\theta^{-1},$$

where $\theta \in \mathbb{R}$ and U_θ is the one-parameter group of unitary operators which rescales particle positions and of photon momenta:

$$x_j \to e^\theta x_j \text{ and } k \to e^{-\theta} k.$$

Resonances

One can show show that:

- Under a certain analyticty condition on coupling functions, the family $H^{SM}_{g\theta}$ has an analytic continuation in θ to the disc $D(0, \theta_0)$, as a type A family in the sense of Kato;
- The real eigenvalues of $H^{SM}_{g\theta},\ \mathrm{Im}\,\theta > 0$, coincide with eigenvalues of H^{SM}_{g} and that complex eigenvalues of $H^{SM}_{g\theta},\ \mathrm{Im}\,\theta > 0$, lie in the complex half-plane $\mathbb{C}^-$;
- The complex eigenvalues of $H^{SM}_{g\theta},\ \mathrm{Im}\,\theta > 0$, are locally independent of θ.

Complex eigenvalues of $H^{SM}_{g\theta},\ \mathrm{Im}\,\theta > 0$, $=$ the *resonances* of H^{SM}_{g}.

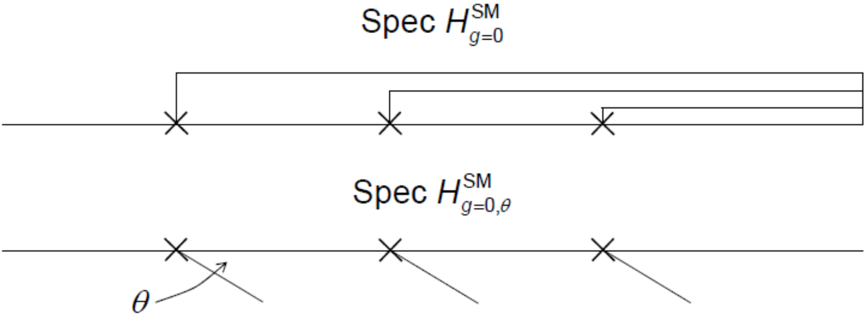
Infrared Problem

The resonances arise from the eigenvalues of the non-interacting Hamiltonian $H^{SM}_{g=0}$. The low energy spectrum of the operator H^{SM}_0 consists of

- Branches $[\epsilon_i^{(p)}, \infty)$ of absolutely continuous spectrum and
- Eigenvalues $\epsilon_i^{(p)}$'s, sitting at the continuous spectrum 'thresholds' $\epsilon_i^{(p)}$'s.

The *absence of gaps* between the eigenvalues and thresholds is a consequence of the fact that the photons are *massless*.
To do perturbation theory in this situation we use - after Bach-Fröhlich-Sigal- the spectral RG.
The problem here is that the perturbation in H^{SM} is *marginal*.
(Bach-Fröhlich-Sigal remove it by additional assumptions.)

Spectrum of $H^{SM}_{g=0}$



Approach

- Perform a new canonical transformation

$$H_g^{SM} \longrightarrow e^{-igF} H_g^{SM} e^{igF}$$

  (a generalized Pauli-Fierz transform), and

- Apply the spectral RG on new – momentum anisotropic – Banach spaces.

These steps allow us to control the RG flow for more singular coupling functions.

Generalized Pauli-Fierz Transformation

Consider one particle of mass be 1. We define the generalized Pauli-Fierz transformation as:

$$H_g^{PF} := e^{-igF(x)} H_g^{SM} e^{igF(x)},$$

where $F(x)$ is the self-adjoint operator on the state space $\mathcal{H}$ given by

$$F(x) = \sum_\lambda \int (\bar{f}_{x,\lambda}(k) a_\lambda(k) + f_{x,\lambda}(k) a^*_\lambda(k)) \frac{d^3k}{\sqrt{|k|}},$$

with the coupling function $f_{x,\lambda}(k)$ chosen as

$$f_{x,\lambda}(k) := e^{-ikx} \frac{\chi(k)}{\sqrt{|k|}} \varphi(|k|^{\frac{1}{2}} e_\lambda(k) \cdot x),$$

$$\varphi \in C^2 \text{ and satisfies } \ \varphi'(0) = 1.$$

Generalized Pauli-Fierz Hamiltonian

Compute: $$H_g^{PF} = \frac{1}{2}(p - gA_1(x))^2 + V_g(x) + H_f + gG(x),$$

where $V(x)$ is a 'nice' potential, $G(x)$ is a 'nice' operator and

$$A_1(x) = \sum_\lambda \int (e^{ikx} a_\lambda(k) + e^{-ikx} a_\lambda^*(k)) \chi_{\lambda,x}(k) \frac{d^3k}{\sqrt{|k|}},$$

$$\chi_{\lambda,x}(k) := e_\lambda(k) e^{-ikx} \chi(k) - \nabla_x f_{x,\lambda}(k).$$

The new coupling function $\chi_{\lambda,x}(k)$ satisfies the estimates $\int \frac{d^3k}{|k|} |\chi_{\lambda,x}(k)|^2 < \infty$ and

$$|\chi_{\lambda,x}(k)| \lesssim \min(1, \sqrt{|k|}\langle x\rangle).$$

The standard Pauli-Fierz transformation:

$$f_{x,\lambda}(k) = \chi(k) e_\lambda(k) \cdot x \implies G = E \cdot x.$$

Renormalization Group

To find the spectral structure of H_θ we use the *spectral renormalization group* (RG):

- Pass from a single operator H_θ to a Banach space $\mathcal{B}$ of generalized Hamiltonians;
- Construct a map, $\mathcal{R}_\rho$, (RG transformation) on $\mathcal{B}$, with the following properties:
  (a) $\mathcal{R}_\rho$ is 'isospectral';
  (b) $\mathcal{R}_\rho$ removes the photon degrees of freedom related to energies $\geq \rho$.
- Relate the dynamics of semi-flow, $\mathcal{R}_\rho^n, n \geq 1$, (called *renormalization group*) to spectral properties of individual operators in $\mathcal{B}$.

RG Map

Consider a pair of orthogonal projections
$\pi_\rho = \chi_{H_f \le \rho}$ and $\overline{\pi}_\rho := \mathbf{1} - \pi_\rho = \chi_{H_f \ge \rho}$.
The *renormalization map* is defined as

$$\mathcal{R}_\rho = \rho^{-1} S_\rho \circ F_\rho,$$

where $\rho > 0$, $S_\rho : \mathcal{B}[\mathcal{H}] \to \mathcal{B}[\mathcal{H}]$ is the *scaling transformation*:

$$S_\rho(\mathbf{1}) \ := \ \mathbf{1}, \qquad S_\rho(a^\#(k)) := \ \rho^{-d/2}\, a^\#(\rho^{-1}k),$$

and F_ρ is the *(smooth) Feshbach-Schur map*,

$$F_\rho(H - \lambda) \ := \ \pi_\rho(H - \lambda \ - \ H\bar{\pi}_\rho(\bar{\pi}_\rho H \bar{\pi}_\rho - \lambda)^{-1}\bar{\pi}_\rho H)\pi_\rho.$$

Isospectrality of F_ρ

The map F_ρ is *isospectral* in the following sense:

(i) $\lambda \in \rho(H) \Leftrightarrow 0 \in \rho(F_\rho(H-\lambda))$;

(ii) $H\psi = \lambda\psi \iff F_\rho(H-\lambda)\,\varphi = 0$, with ψ and φ related in a constructive way;

(iii) $\dim \mathrm{Ker}(H-\lambda) = \dim \mathrm{Ker} F_\rho(H-\lambda)$;

(iv) $(H-\lambda)^{-1}$ exists $\Leftrightarrow$ $F_\rho(H-\lambda)^{-1}$ exists and

$$(H-\lambda)^{-1} = Q_\pi(\lambda)\, F_\rho(H-\lambda)^{-1}\, Q_\pi(\lambda)^{\#}$$

for some well-defined operators $Q_\pi(\lambda)$ and $Q_\pi(\lambda)^{\#}$.

A Banach Space of Hamiltonians

Operators on the Fock space $\mathcal{F}$ are in the generalized normal form if they can be written as:

$$W = \sum_{m,n} \int_{B_1^{m+n}} \prod_{i=1}^{m+n} d^3 k_i \prod_{i=1}^{m} a^*(k_i)\, w_{m,n}(H_f; k) \prod_{i=m+1}^{m+n} a(k_i),$$

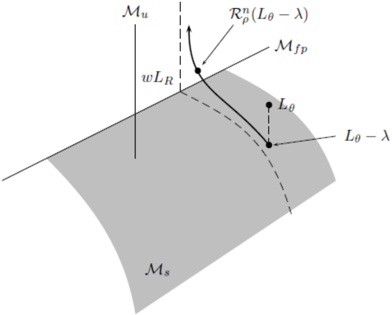
$B_1^r := \otimes_1^r B_1$, B_1 is the unit ball in $\mathbb{R}^3$ and $k := (k_1, \dots, k_{m+n})$. The functions $w_{m,n}(r,k) \in C^2$ are symmetric w. r. t. the variables $(k_1, \dots, k_m)$ and $(k_{m+1}, \dots, k_{m+n})$ and obey

$$\|w_{m,n}\|_\mu \;:= \sum_{n=0}^{s} \max_j \sup_{r \in I, k \in B_1^{m+n}} \left| |k_j|^{-\mu} \prod_{i=1}^{m+n} |k_i|^{1/2} \partial_r^n w_{m,n}(r;k) \right|,$$

where $\mu > 0$, $s \geq 0$ and $I := [0,1]$.

RG Dynamics

The flow, $\mathcal{R}_\rho^n$, has the fixed-point manifold $\mathcal{M}_{fp} := \mathbb{C}H_f$, unstable manifold $\mathcal{M}_u := \mathbb{C}\mathbf{1}$, and (complex) co-dimension 1 stable manifold $\mathcal{M}_s$ for $\mathcal{M}_{fp}$.



Stable and unstable manifolds.

RG and Spectral Properties

Adjust the parameter λ, so that $H_\theta - \lambda$ is in a ρ^n-neighborhood of the stable manifold $\mathcal{M}_s$

$\Longrightarrow H_\theta - \lambda$ is in the domain of $\mathcal{R}_\rho^n$.

$\Longrightarrow H^{(n)}(\lambda) := \mathcal{R}_\rho^n(H_\theta - \lambda) \approx wH_f$, for some $w \in \mathbb{C},\ \mathrm{Re}\ w > 0$, and n sufficiently large

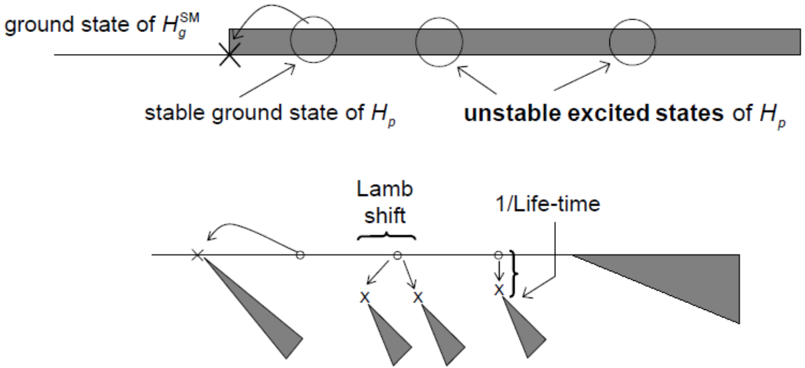
$\Longrightarrow$ Spectral information about $H^{(n)}(\lambda)$.

$\Longrightarrow$ Spectral information about $H^{(n-1)}(\lambda)$ (by 'isospectrality' of $\mathcal{R}_\rho$)

...

$\Longrightarrow$ Spectral information about H_θ.

Migration of EVs of H_p



Main Results I

Theorem

Fix $\nu < \inf \sigma_{ess}(H_p)$ *and let* $g \ll \epsilon^{(p)}_{gap}(\nu)$*. Then for* $g \neq 0$,

- *Eigenvalues,* $\epsilon^{(p)}_j < \nu$, *of* $H^{SM}_{g=0} \implies$ *resonance and/or bound state eigenvalues,* $\epsilon_{j,k}$, *of* H^{SM}_g;
- $\epsilon_{j,k} = \epsilon^{(p)}_j + O(g^2)$ *and the total multiplicity of* $\epsilon_{j,k}$ $\forall k$ *equals the multiplicity of* $\epsilon^{(p)}_j$;
- H^{SM}_g *has a ground state, originating from a ground state of* $H^{SM}_{g=0}$;
- $\epsilon_{j,k}$*'s are independent of* θ.

Main Results II

Let $\epsilon_0 := \inf \sigma(H_g^{SM})$ be the ground state energy of H_g^{SM} and let

$$S_{j,k} := \{z \in \mathbb{C} \mid \frac{1}{2}\mathrm{Re}\,(e^{\theta}(z - \epsilon_{j,k})) \geq |\mathrm{Im}\,(e^{\theta}(z - \epsilon_{j,k}))|\}.$$

Theorem

Assume $g \ll \epsilon_{gap}^{(p)}(\nu)$. *Then for a dense set of vectors* Ψ *and* Φ*,*

- *the matrix elements* $F(z, \Psi, \Phi) := \langle \Psi, (H_g^{SM} - z)^{-1}\Phi \rangle$ *of the resolvent of* H_g^{SM} *have meromorphic continuations from* $\mathbb{C}^+$ *across the interval* (ϵ_0, ν) *of the essential spectrum of* H_g^{SM} *into the domain* $\{z \in \mathbb{C}^- | \ \epsilon_0 < \mathrm{Re}\, z < \nu\}$*, with the wedges* $S_{j,k},\ 0 \leq j \leq j(\nu)$*, deleted;*
- *this continuation has poles at* $\epsilon_{j,k}$ *in the sense that* $\lim_{z \to \epsilon_{j,k}} (\epsilon_{j,k} - z) F(z, \Psi, \Phi)$ *is finite and, for a finite-dimensional subspace of* Ψ*'s and* Φ*'s, nonzero.*

Resonance Poles

Theorem (Abou Salem-Faupin-Fröhlich-Sigal):
For each Ψ and Φ from a dense set, the meromorphic continuation, $F(z, \Psi, \Phi)$, of the matrix element $\langle \Psi, (H_g^{SM} - z)^{-1} \Phi \rangle$, near the resonance $\epsilon_{j,k}$ of H_g^{SM}, is of the following form:

$$F(z, \Psi, \Phi) = (\epsilon_{j,k} - z)^{-1} p(\Psi, \Phi) + r(z, \Psi, \Phi).$$

Here p and $r(z)$ are sesquilinear forms in Ψ and Φ with $r(z)$, analytic in $z \in Q := \{z \in \mathbb{C}^- | \ \epsilon_0 < \mathrm{Re}\, z < \nu\} / \bigcup_{j \leq j(\nu), k} S_{j,k}$ and bounded on the intersection of a neighbourhood of $\epsilon_{j,k}$ with Q as

$$|r(z, \Psi, \Phi)| \leq C_{\Psi,\Phi} |\epsilon_{j,k} - z|^{-\gamma} \text{ for some } \gamma < 1.$$

Moreover, $p \neq 0$ at least for one pair of vectors Ψ and Φ and $p = 0$ for a dense set of vectors Ψ and Φ in a finite co-dimension subspace. The multiplicity of a resonance is the rank of the residue at the pole.

Open Problems

- Resonances $\Longleftrightarrow$ poles of the scattering matrix;

- Minimal and maximal velocity of photons

  (Sigal-Soffer : $\|d\Gamma(\chi_{|y|\geq t^{\gamma}})\psi_t\| \to 0, \quad \gamma > 1, \ y =$ photon coordina

  $< \psi_t, N\psi_t > \leq C$;

- Asymptotic completeness.

Thank-you for your attention.