

An analysis of lattice KPZ equation

T. Sasamoto H. Spohn

TU-München

11 Jun 2009

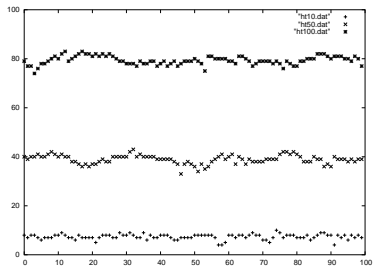
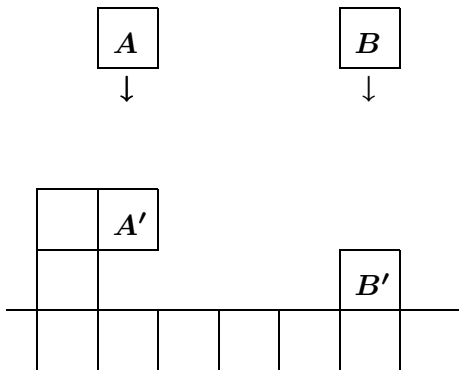
1D surface growth

- Paper wetting, bacteria colony, crystal growth
- Non-equilibrium statistical mechanics
- Stochastic processes
- Exactly solvable models
- Classical, stochastic, 1 dimension



Simulation models

Ex: ballistic deposition model



Fluctuations of surface

$h(x, t)$: surface height at position x and at time t

Scaling (L : system size)

$$W(L, t) = \langle (h(x, t) - \langle h(x, t) \rangle)^2 \rangle^{1/2} = L^\alpha \Psi(t/L^z)$$

For $t \rightarrow \infty$

$$\Psi(y) \sim \text{Const} \quad \Rightarrow \quad W(L, t) \sim L^\alpha$$

For $t \sim 0$

$$\Psi(y) \sim y^\beta \quad \Rightarrow \quad W(L, t) \sim t^\beta$$

where $\alpha = \beta z$

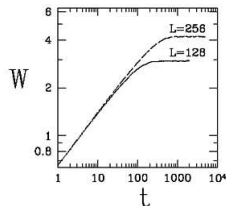


Figure 1. Interface width W versus time t for the RSOS (Ref. [11]) in 1 + 1 dimensions, in two different lattice lengths.

In many models, $\alpha = 1/2, \beta = 1/3$

Kardar-Parisi-Zhang(KPZ) equation

1986 Kardar Parisi Zhang

$$\partial_t h(x, t) = \frac{1}{2} \lambda_0 (\partial_x h(x, t))^2 + \nu_0 \partial_x^2 h(x, t) + \sqrt{D_0} \xi(x, t)$$

where $\langle \xi(x, t) \xi(x', t') \rangle = \delta(x - x') \delta(t - t')$

Dynamical RG analysis $\rightarrow \alpha = 1/2, \beta = 1/3$

KPZ universality class

The slope $u = \partial_x h$ satisfies the noisy Burgers equation

$$\partial_t u = \frac{1}{2} \lambda_0 \partial_x u^2 + \nu_0 \partial_x^2 u + \sqrt{D_0} \partial_x \xi$$

Recent developments

Recently many results have been obtained for **other** stochastic growth processes in the KPZ universality class

- Polynuclear growth(PNG) models, asymmetric simple exclusion process (ASEP),...
- Exact solutions using Bethe ansatz, random matrix theory
- Exponents, height distributions, two point functions,...

Few results for the KPZ equation itself.

The noise in the noisy Burgers equation is singular.

- Smoothing the noise
- Discretizing the equation (our choice here)

Lattice KPZ equation

$$\begin{aligned}\frac{d}{dt}u_j &= \frac{1}{6}\lambda_0(u_{j+1}^2 + u_j u_{j+1} - u_{j-1} u_j - u_{j-1}^2) \\ &\quad + \nu_0(u_{j+1} - 2u_j + u_{j-1}) + D_0^{1/2}(\xi_j - \xi_{j-1})\end{aligned}$$

where $j \in \mathbb{Z}$ and $\{\xi_j\}$ is independent white noise
Space is discrete; time and slope are continuous

Why this discretization?

- 1 Only nearest neighbors
- 2 Explicit stationary measures

Current function

$$w_j = \frac{1}{6}\lambda_0(u_j^2 + u_j u_{j+1} + u_{j+1}^2) + \nu_0(u_{j+1} - u_j)$$

Then

$$\frac{d}{dt}u_j = w_j - w_{j-1} + \sqrt{D_0}(\xi_j - \xi_{j-1}), \quad j \in \mathbb{Z}$$

The quantity of our interest: stationary two-point function

$$S(j, t) = \langle u_j(t)u_0(0) \rangle - \langle u_0(0) \rangle^2$$

where $\langle \dots \rangle$ is the average wrt stationary measure

A remark on KPZ Scaling

- Universal and model-dependent constants
- Given (not necessarily for lattice KPZ)
 - Growth velocity $v(\rho)$ for slope ρ . Set $\lambda_0 = v''(\rho)$
 - Roughness amplitude $\sum_j S(j, 0) := \chi$

the scale in the height and space directions are automatically determined to be $(\frac{1}{2}\lambda_0\chi^2t)^{1/3}$ and $(2\lambda_0^2\chi)^{1/3}t^{2/3}$.

- Two point function

$$S(j, t) \stackrel{t \rightarrow \infty}{\cong} \chi(2\lambda_0^2\chi t^2)^{-1/3} f_{\text{KPZ}}((2\lambda_0^2\chi t^2)^{-1/3}j)$$

- Knowledge of the stationary measure is enough for the computations of these quantities.

The generator (on a ring $[1, \dots, N]$)

$$L_N = \sum_{j=1}^N [(w_j - w_{j-1})\partial_j + \frac{1}{2}D_0(\partial_j - \partial_{j-1})^2]$$

Stationary measures: independent Gaussians ($\rho \in \mathbb{R}$)

$$\prod_{j=1}^N \left\{ \frac{1}{Z} \exp[-(\nu_0/D_0)(u_j - \rho)^2] \right\} =: (\psi_{0,\rho}^N(u))^2$$

Growth velocity and roughness amplitude

$$v(\rho) = \frac{\lambda_0}{6} \left(\frac{D_0}{\nu_0} + 3\rho^2 \right)$$

$$\chi = \frac{D}{2\nu_0}$$

In the following we only consider the case $\rho = 0$.

Simulation (for a slightly different discrete version, size $L = 1024$)

* discretized KPZ from Katzav, Schwartz, PRE 2004

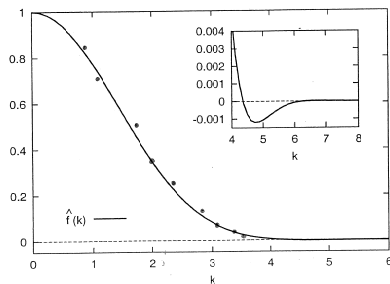


Figure 4: The Fourier transform $\hat{f}(k)$ of the scaling function f exact solution

Hamiltonian: ground state transformation

$$\psi_{0,0}^N L_N (\psi_{0,0}^N)^{-1} = -H_N$$

$$H_N = \frac{1}{2} \sum_{j=1}^N (D_0 (\partial_j - \partial_{j-1}))^2 + \frac{\nu_0^2}{D_0} (u_j - u_{j-1})^2 - \sum_{j=1}^N (\tilde{w}_j - \tilde{w}_{j-1}) \partial_j$$

where \tilde{w}_j is the nonlinear part of w_j .

At this point we take $N \rightarrow \infty$.

Field theoretic formulation

Annihilation/creation operators at site j

$$a_j = \frac{1}{\sqrt{2}}(\alpha u_j + \alpha^{-1} \partial_j), \quad a_j^* = \frac{1}{\sqrt{2}}(\alpha u_j - \alpha^{-1} \partial_j),$$

with $\alpha = (\nu_0/D_0)^{1/2}$. They satisfy $[a_i, a_j^*] = \delta_{ij}$.

Setting $\nu_0 = 1$, $\lambda = \frac{1}{3 \cdot 2^{5/2}} (\nu_0/D_0)^{1/2} \lambda_0$, we get

$$H = H_0 + \lambda(A^* - A)$$

with

$$H_0 = \sum_{j \in \mathbb{Z}} (a_{j+1} - a_j)^* (a_{j+1} - a_j)$$

$$A = \sum_{j \in \mathbb{Z}} (a_j a_{j+1}^* a_{j+1} + a_j a_j a_{j+1}^* - a_j^* a_{j+1} a_{j+1} - a_j^* a_j a_{j+1})$$

Fock Space representation

Fock space

$$\mathfrak{F} = \bigoplus_{n=0}^{\infty} \mathfrak{F}_n$$

where $\mathfrak{F}_n = (\ell_2(\mathbb{Z}))_{\text{sym}}^{\otimes n}$ is the n -particle subspace.

An element $f \in \mathfrak{F}$ is the sequence $\{f_0, f_1, \dots, f_n, \dots\}$.

$f_n : \mathbb{Z}^n \rightarrow \mathbb{C}$ and f_n is symmetric in its arguments, $f_0 \in \mathbb{C}$.

The scalar product on \mathfrak{F}_n is

$$\langle f, g \rangle = \sum_{(x_1, \dots, x_n) \in \mathbb{Z}^n} f_n(x_1, \dots, x_n)^* g_n(x_1, \dots, x_n)$$

The Fock vacuum is the vector $\Omega = (1, 0, 0, \dots)$.

Fourier transformation

$$\widehat{f}(k) = \sum_{j \in \mathbb{Z}} e^{-i2\pi k j} f(j)$$

$$H_0 = \int_{\mathbb{T}} dk \omega(k) a(k)^* a(k), \quad \omega(k) = 2(1 - \cos(2\pi k))$$

and

$$A = i \int_{\mathbb{T}^3} dk_1 dk_2 dk_3 \delta(k_1 - k_2 - k_3) (2 \sin(2\pi(k_1)) \\ + \sin(2\pi k_2) + \sin(2\pi k_3)) a(k_1)^* a(k_2) a(k_3)$$

Notice

$$H_0 : \mathfrak{F}_n \rightarrow \mathfrak{F}_n$$

$$A : \mathfrak{F}_{n+1} \rightarrow \mathfrak{F}_n$$

Two point function

$$S(j, t) = \langle \Omega, a_j e^{-tH} a_0^* \Omega \rangle$$

$$\stackrel{t \rightarrow \infty}{\cong} \chi(2\lambda_0^2 \chi t^2)^{-1/3} f_{\text{KPZ}}((2\lambda_0^2 \chi t^2)^{-1/3} j)$$

Variance

$$\text{Var}(t) = \sum_{j \in \mathbb{Z}} j^2 S(j, t) \stackrel{t \rightarrow \infty}{\cong} \chi(2\lambda_0^2 \chi t^2)^{2/3} \int dx x^2 f_{\text{KPZ}}(x)$$

with $\int dx x^2 f_{\text{KPZ}}(x) = 0.510523 \dots$ (computed for PNG, ASEP)

Zero total momentum space

Total momentum operator

$$P = 2\pi \int_{\mathbb{T}} dk k a(k)^* a(k)$$

Since H is translation invariant, $[H, P] = 0$

Direct integral decompositions corresponding to P

$$\mathfrak{F} = \int_{\mathbb{T}}^{\oplus} dk \mathfrak{F}(k), \quad H = \int_{\mathbb{T}}^{\oplus} dk H(k)$$

Due to the summation \sum_j in $\text{Var}(t) = \sum_{j \in \mathbb{Z}} j^2 S(j, t)$, we can restrict our discussions to $\mathfrak{F}(0)$.

The scalar product in this subspace will be denoted by $\langle f, g \rangle_0$.

Laplace transform

For $\zeta > 0$, using the Green-Kubo formula

$$\begin{aligned}\widehat{\text{Var}}(\zeta) &= \int_0^\infty dt e^{-\zeta t} \text{Var}(t) \\ &= \frac{1}{2} \chi \zeta^{-2} + \zeta^{-2} \langle \hat{w}_0, (\zeta + H)^{-1} \hat{w}_0 \rangle_0\end{aligned}$$

Here \hat{w}_0 is the Fourier transform of the current function at the origin and only the second component is nonzero,

$$\hat{w}_0 = (0, 0, \hat{w}_{0,2}, 0, \dots)$$

When $\zeta \rightarrow 0$ we expect from the KPZ scaling

$$\langle \hat{w}_0, (\zeta + H)^{-1} \hat{w}_0 \rangle_0 \stackrel{\zeta \rightarrow 0}{\cong} \zeta^{-1/3} \Gamma(7/3) \chi (2\lambda^2 \chi)^{2/3} \int dx x^2 f_{\text{KPZ}}(x)$$

Resolvent hierarchy

2004 Landim Quastel Salmhofer Yau

To study the asymptotics of $\langle \hat{w}_0, (\zeta + H)^{-1} \hat{w}_0 \rangle_0$, we consider the resolvent equation

$$(\zeta + H)u_\zeta = \hat{w}_0, \quad H = H_0 + \lambda(A^* - A)$$

This equation has the structure

$$\begin{bmatrix} \zeta + H_0 & \lambda A^* & \mathbf{0} & \dots \\ -\lambda A & \zeta + H_0 & \lambda A^* & \mathbf{0} \\ \mathbf{0} & -\lambda A & \zeta + H_0 & \lambda A^* & \mathbf{0} \\ \vdots & \ddots & \ddots & & \end{bmatrix} \begin{bmatrix} u_{\zeta,2} \\ u_{\zeta,3} \\ u_{\zeta,4} \\ \vdots \end{bmatrix} = \begin{bmatrix} \hat{w}_{0,2} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \end{bmatrix}$$

Truncation up to n th components gives alternating bounds,

$$b_3 \leq b_5 \leq \dots \leq b_4 \leq b_2, \quad \langle \hat{w}_0, (\zeta + H)^{-1} \hat{w}_0 \rangle_0 = \lim_{n \rightarrow \infty} b_n$$

Bounds

Theorem

In the limit $\zeta \rightarrow 0$, the following bounds are valid

$$\lambda 2^{-5/4} 3^{-1/2} \zeta^{-1/4} \leq \langle \hat{w}_0, (\zeta + H(0))^{-1} \hat{w}_0 \rangle_0 \leq \frac{9\lambda^2}{2^{5/2}} \zeta^{-1/2}$$

Proof of the upper bound ($n = 2$): It holds

$$\begin{aligned} \langle \hat{w}_0, (\zeta + H)^{-1} \hat{w}_0 \rangle_0 &\leq \langle \hat{w}_0, (\zeta + H_{0,2})^{-1} \hat{w}_0 \rangle_0 \\ &= \lambda^2 \int_{\mathbb{T}} dk_1 \frac{1}{2} (2 + \cos(2\pi k_1))^2 (\zeta + 2\omega(k_1))^{-1} \cong \frac{9\lambda^2}{2^{5/2}} \zeta^{-1/2} \end{aligned}$$

for $\zeta \rightarrow 0$ where $H_{0,2}$ is H_0 projected on \mathfrak{F}_2 .

The lower bound ($n = 3$) is more involved (similar to LQSY).

Approximation

The bound for truncation up to $n = 4$ reads

$$\langle \hat{w}_0, \left\{ \zeta + H_0 + \lambda^2 A(\zeta + H_0 + \lambda^2 A(\zeta + H_0)^{-1} A^*)^{-1} A^* \right\}^{-1} \hat{w}_0 \rangle_0$$

We approximate the action of $A(\zeta + H_0)^{-1} A^*$ by a multiplication

$$A(\zeta + H_0)^{-1} A^* \sim V_l$$

Then the bound above can be computed (within this approximation).

The same approximation scheme works for general n .

Set

$$\alpha_n = \frac{1}{3} \left(1 - \left(-\frac{1}{2} \right)^{n-1} \right) = \frac{1}{2}, \frac{1}{4}, \frac{3}{8}, \dots$$

Under this approximation scheme, we get

$$b_n \cong \frac{|\hat{w}_{0,2}|^2}{2 \left(\frac{\chi \lambda_0^2}{16} \right)^{\alpha_{n-1}} (2\zeta)^{\alpha_n}}$$

Hence

$$\langle \hat{w}_0, (\zeta + H)^{-1} \hat{w}_0 \rangle_0 = \lim_{n \rightarrow \infty} b_n \cong C \zeta^{-1/3}$$

The true KPZ exponent $\frac{1}{3}!!$

For the 2nd moment, we get the value 0.5291... This is fairly close to but not the same as the exact value 0.5105...

The approximation needs to be further improved.

Summary and future problems

Summary

- Lattice version of the KPZ equation
- Stationary two point function, its variance
- Rigorous bounds
- Non-rigorous approximation leading to KPZ exponents

Future problems

- Establishing KPZ scaling
- Other initial and boundary conditions
- Bethe ansatz, determinantal structures?