Does existence and regularity of the sol.ns to the Boltzmann eq.n imply the global validity?

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Classical Boltzmann eq.n for hard spheres

$$(\partial_t + v \cdot \nabla_x) f = Q(f, f)$$

$$Q = \int dv_1 \int_{S^2_+} d\omega \, B(\omega, v - v_1)(f' f'_1 - f f_1).$$

$$B(\ldots) = \omega \cdot (v - v_1); \qquad S^2_+ = \{\omega \in S^2 | \omega \cdot (v - v_1) \geq 0\}$$

$$f = f(v), f_1 = f(v_1), f' = f(v'), f'_1 = f(v'_1)$$

$$v' = v - \omega \cdot (v - v_1)\omega, \quad v'_1 = v_1 + \omega \cdot (v - v_1)\omega.$$

Scaling limits: Low-density

X, T microscopic variables, x, t macroscopic variables. ε a scale parameters (e.g. the inverse of the n. of Angstrom in a meter). $x = \varepsilon X$, $t = \varepsilon T$. Write the fundamental eq.ns in macro variables. Set $N = \varepsilon^{-2}$ (low density or BG). One particle: binary coll.ns. n. of coll.ns per unit time is finite.

Boltzmann eq.n for H-S. Lanford '75, Illner-P. '89

Other scalings

$x = \varepsilon X$, $t = \varepsilon^{(1-\alpha)} T$, $1 > \alpha > 0$. Write the fundamental eq.ns in macro variables. Setting $N = \varepsilon^{-(2+\alpha)}$, we derive the homogeneous B eq.n. (at least for short times)

For $\alpha = 1$ recover the hydrodynamic limit, while, for $\alpha = 0$ you get the low density limit. Indeed the one-particle distribution:

$$(\partial_t + \varepsilon^{\alpha} v \cdot \nabla_x) f^{\varepsilon} \approx Q(f^{\varepsilon}, f^{\varepsilon}).$$

Use Lanford (x becomes a parameter) to get the homogeneous B eq.n (even for not traslationary inveriant states) for short times.

Physical significance and stationary problem.

Assuming global smooth solutions to the B eq.n (e.g. homogeneous, near the equilibrium...), does the B-G limit hold in the large?

Work in progress.

Semidiscrete model in 2-D.

Boltzmann equation for hard-disks

$$(\partial_t + v \cdot \nabla_x) g(x, v; t) = Q(g, g)(x, v; t)$$

$$Q(g,g)(x,v;t) = \int dv_1 \int_{S^1_+} d\omega \omega \cdot (v - v_1)$$

$$[g(x,v')g(x,v_1') - g(x,v)g(x,v_1))],$$

where

$$S^1_+ = \{\omega \in S^1 | (v - v_1) \cdot \omega \geq 0\}$$

selects the set of incoming velocities.

Collision law:

$$v' = v - 2(\omega \cdot v)\omega, \quad v_1' = v_1 + 2(\omega \cdot v_1)\omega.$$

One-particle phase space

$$(x, v) \in T^2 \times S^1$$

T^2 is the 2$-$D torus. The velocity $\in S^1$.

Energy is conserved, but not the momentum.

We also consider the following Boltzmann-Enskog equation (use it as a bribge):

$$(\partial_t + v \cdot \nabla_x) g_\varepsilon(x, v; t) = Q_\varepsilon(g_\varepsilon, g_\varepsilon)(x, v; t)$$

where

$$Q_\varepsilon(g, g)(x, v; t) = \int dv_1 \int_{S^1_+} d\omega \omega \cdot (v - v_*)$$

$$[g(x, v')g(x - \varepsilon\omega, v'_*) - g(x, v)g(x + \varepsilon\omega, v_*))],$$

Consider the initial value problem, with the same initial value

$$g_0 : T^2 \times S^1.$$

g_0 is such that there exists unique solutions $g(x, v; t)$ and $g_\varepsilon(x, v; t)$, up to time $T > 0$, such that

$$\sup_{x,v,t \le T} g_\varepsilon(x, v; t) = G < +\infty; \qquad \sup_{x,v,t \le T} g(x, v; t) = G.$$

Moreover:

$$\lim_{\varepsilon \to 0} g_\varepsilon(x, v; t) = g(x, v; t),$$

in $C(T^2 \times S^1 \times [0, T])$.

Particles

N-particles of diameter $\varepsilon > 0$.

$Z_N = (X_N, V_N) = (z_1 \dots z_N) = (x_1, v_1 \dots x_N, v_N)$ $x_i \in T^2$ and $v_i \in S^1$, with the hard-core condition $|x_i - x_j| > \varepsilon$ for all $i \neq j$. $Z_N \to \Phi^t(Z_N)$ the dynamical flow with the above collision rule.

Boltzmann-Grad limit:

$$N \to \infty, \varepsilon \to 0, N\varepsilon = 1.$$

Initial distribution

$$W_0^N(Z_N) = \prod_{i=1}^{N} g_0(z_i) \prod_{i<j} \chi_{i,j}^{\varepsilon} Norm$$

$$\chi_{i,j}^{\varepsilon} = \chi(\{|x_i - x_j| > \varepsilon\})$$

Hierarchies

Let $g_\varepsilon(t)$ be the sol.n to the B-E eq.n equation. Consider the products:

$$g_j^\varepsilon(X_j, V_j; t) = g_\varepsilon^{\bigotimes j}(X_j, V_j; t), \qquad j = 1 \ldots \infty.$$

Then $g_j^\varepsilon(t)$ solves the hierarchy of equations

$$(\partial_t + \sum_{k=1}^{j} v_k \cdot \nabla_{x_k}) g_j^\varepsilon = C_{j+1} g_{j+1}^\varepsilon,$$

where

$$C_{j+1} = \sum_{k=1}^{j} C_{k,j+1}$$

and

$$C_{k,j+1} g_{j+1}(x_1 \dots x_j; v_1 \dots v_j) =$$

$$\int dv_{j+1} \int_{S^2_+} d\omega \omega \cdot (v_k - v_{j+1})$$

$$[g_{j+1}(x_1 \dots x_j, x_k + \varepsilon\omega; v_1 \dots v'_k \dots v'_{j+1}) -$$

$$g_{j+1}(x_1 \dots x_j, x_k - \varepsilon\omega; v_1 \dots v_k \dots v_{j+1})].$$

We represent the solution by means of the Duhamel expansion:

$$g_j^N(t) = \sum_{n=0}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n$$

$$S(t-t_1)C_{j+1}^{\varepsilon} \ldots S(t_{n-1}-t_n)C_{j+n}^{\varepsilon} S^{\varepsilon}(t_n) g_{n+j}^0.$$

where

$$g_{n+j}^0 = g_0^{\otimes(n+j)}$$

and

$$S(t)g_j(X_j, V_j) = g_j(X_j - V_j t, V_j).$$

An analogous expansion holds for the marginals of the time evolved measure

$$W^N(Z_N; t)$$

denoted by $f_j^\varepsilon(t)$. See later on.

It is convenient to represent each term of the Duhamel expansion in terms af integrals of the initial datum. We call a n- collision, j-particle tree a collection of integers $k_1 \dots k_n$ s.t.:

$$k_1 \in I_j, k_2 \in I_{j+1}, \dots k_n \in I_{j+n}$$

where $I_j = \{1, 2, \ldots j\}$.

Next set

$$\mathbf{t}_n = (t_1, \ldots t_n),$$

$$\omega_{\mathbf{n}} = \omega_1 \ldots \omega_n$$

$$\mathbf{V}_{j,n} = v_{j+1} \ldots v_{j+n}$$

$$\sigma = \sigma_1 \ldots \sigma_n; \quad \sigma_j = \pm 1$$

and define

$$d\Lambda(\mathbf{t}_n, \omega_n, \mathbf{V}_{j,j+n}) = dt_1 \ldots dt_n \chi(\{t_1 > t_2 \ldots > t_n\})$$

$$d\omega_1 \dots d\omega_n dv_{j+1} \dots dv_{j+n}$$

$$g_j^\varepsilon(Z_j;t) = \sum_{n=0}^{\infty} \sum_{G(j,n)} \sum_{\sigma} (-1)^{|\sigma|}$$

$$\int d\Lambda(\mathbf{t}_n, \omega_n, \mathbf{V}_{j,j+n})\, B_n\, g^0_{j+n}(\tilde{\zeta}^\varepsilon_{j+n}(0)).$$

where $B_n = \prod B$ and, for $s \in (0,t)$

$$\tilde{\zeta}^\varepsilon(s) = (\tilde{\xi}^\varepsilon(s), \tilde{\eta}^\varepsilon(s))$$

is the backward flow of the positions and velocities of the particles, created according to $G(j,n), \sigma, (\mathbf{t}_n, \omega_n, \mathbf{V}_{j,j+n})$. Without recollisions.

Short time bound

$$\sum_{G(j,n)} = j(j+1)\dots(j+n-1);$$

$$\int d\Lambda B_n \le \frac{C_0^n}{n!}t^n;$$

$$g^0_{j+n} \le G^{j+n}$$

hence

$$\sum_n (\dots) \le (C_0 G)^j \sum_n (C_0 G t)^n$$

Compare

$$g_j^\varepsilon(Z_j;t) = \sum_{n=0}^{\infty} \sum_{G(j,n)} \sum_\sigma (-1)^{|\sigma|}$$

$$\int d\Lambda(\mathbf{t}_n, \omega_n, \mathbf{V}_{j,j+n})\, B_n^\varepsilon\, g_{j+n}^0(\tilde{\zeta}_{j+n}^\varepsilon(0)).$$

$$f_j^\varepsilon(Z_j;t) = \sum_{n=0}^{N} \sum_{G(j,n)} \sum_\sigma (-1)^{|\sigma|} \prod_l \alpha_l$$

$$\int d\Lambda(\mathbf{t}_n, \omega_n, \mathbf{V}_{j,j+n})\, B_n^\varepsilon\, f_{j+n}^{\varepsilon,0}(\zeta_{j+n}^\varepsilon(0)).$$

$\zeta^\varepsilon(s)$ is the backward flow WITH recollisions.
$\alpha_l = O(1)$.

The main difference is that

$$f^{\varepsilon,0}_{j+n}(\zeta^\varepsilon_{j+n}(0)) \neq g^0_{j+n}(\tilde{\zeta}^\varepsilon_{j+n}(0))$$

due to the recoll.ns. Small $d\Lambda$ measure.

Try to iterate. You can prove

$$f^\varepsilon_j(Z_j;t) = g^\varepsilon_j(Z_j;t) + E^\varepsilon_j(Z_j;t)$$

$$|E^\varepsilon_j(Z_j;t)| \leq B^j \varepsilon^\gamma$$

outside the backward overlapping. Irreversibility.

The difficulty of the iteration is two-fold. The estimate is not uniform. The error, even if small, propagates. No advantage of the uniform estimate on g_ε.

$I, J, K....$ subset of indices. $|I|$ cardinality of I.

Seek for an expansion of the form (ansatz)

$$f_I^\varepsilon(Z_I, t) = \sum_{K \subset I} g_K^\varepsilon(Z_K, t) R(Z_{I/K}, t)$$

where $g_K^\varepsilon(Z_K, t) = \prod_{k \in K} g^\varepsilon(z_k; t)$ and

$$|R(Z_J, t)| \le \varphi_\varepsilon(|J|)$$

$$\varphi_\varepsilon(j) \to 0; \qquad \varepsilon \to 0.$$

Then, assuming (1) at time t, outside the backward overlapping manifold, then prove that it holds also at time $t + \tau$ (τ very small but fixed).

To see the main idea consider at time t only the leading term $g_K^\varepsilon(Z_K, t)$.

Then ($|I| = j$)

$$f_I^{\varepsilon}(Z_I; t+\tau) = \sum_{n=0}^{N} \sum_{G(j,n)} \sum_{\sigma} (-1)^{|\sigma|}$$

$$\int d\Lambda(\mathbf{t}_n, \omega_n, \mathbf{V}_{j,j+n})\, B_n^{\varepsilon}\, g_{S(I)}^{\varepsilon}(\zeta^{\varepsilon}(0), t) =$$

$$\sum_{n_1,\dots n_j} \sum_{G(n_1)\dots G(n_j)} \sum_{\sigma_1 \dots \sigma_{\mathbf{j}}} (-1)^{\sum |\sigma_{\mathbf{l}}|} \sum_{K \subset I}$$

$$\int d\Lambda_{n_1} B \dots \int d\Lambda_{n_j} B \quad \bar{\chi}_K^{rec} \chi_{I/K}^{rec}$$

$$g_{S(K)}^{\varepsilon}(\tilde{\zeta}^{\varepsilon}(0), t)\, g_{S(I/K)}^{\varepsilon}(\zeta^{\varepsilon}(0), t).$$

But

$$\bar{\chi}_K^{rec} = \sum_{L \subset K} \Gamma_{K/L}$$

hence

$$f_I^\varepsilon(Z_I; t+\tau) = \sum_{L \subset I} g^T(Z_L; t+\tau) R(Z_{I/L}; t+\tau)$$

All the trees I/L recollide, so that their $d\Lambda$ measure is small. We expect

$$|R(Z_r; t+\tau)| \leq \varepsilon^{\gamma r} r!$$