Quasi-potential for Burgers equation

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Boundary Driven WASEP

State Space: Fix $N \geq 1$.

$$\{0,1\}^{\{1,\ldots,N-1\}} \qquad \eta = (\eta(1), \ldots, \eta(N-1))$$

Bulk dynamics: $1 \leq x \leq N-2$

- $x \to x+1$ at rate $\alpha + N^{-1}$
- $x+1 \to x$ at rate α

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Boundary dynamics: Fix $0 \leq \rho_0 < \rho_1 \leq 1$.

- Right boundary: creation at rate ρ_1, annihilation at rate $(1-\rho_1)$
- Left boundary: creation at rate ρ_0, annihilation at rate $(1-\rho_0)$

Stationary state

Stationary state: μ_α^N (depends on ρ_0, ρ_1)

- $\varphi = \log(\rho/1-\rho)$ chemical potential.
- $\alpha_c = [\varphi(1) - \varphi(0)]^{-1}$. No current, process reversible, μ_α^N is product measure.
- $\alpha > \alpha_c$. Current from right boundary to the left.
- $\alpha < \alpha_c$. Current from left boundary to the right.

Nonequilibrium free energy

$$\pi^N(\cdot) \;=\; \frac{1}{N}\sum_{x=1}^{N-1} \eta(x)\,\delta(\cdot - x/N)$$

$$\lim_{N\to\infty} \mu_\alpha^N\left[\pi^N \sim \overline{\rho}_\alpha\right] \;=\; 1$$

$$\begin{cases} 0 = \alpha\rho_{xx} \;-\; f(\rho)_x \\ \rho(0) = \rho_0\,, \quad \rho(1) = \rho_1\,. \end{cases} \qquad f(\rho) \;=\; \rho\,(1-\rho)$$

$$\mu_\alpha^N\left[\pi^N \sim \rho\right] \approx e^{-NS(\rho)}\,,$$

S nonequilibrium free energy

Equilibrium free energy

- $\alpha = \alpha_c$.

$$S(\rho) \;=\; \int_0^1 \left\{ \rho(x) \log \frac{\rho(x)}{\overline{\rho}_\alpha(x)} \;+\; [1-\rho(x)] \log \frac{1-\rho(x)}{1-\overline{\rho}_\alpha(x)} \right\} dx$$

$\alpha > \alpha_c$

Enaud and Derrida (04), **Derrida, Lebowitz and Speer** (01)

$$S(\rho) = \int_0^1 \Big\{ h(\rho) + (1-\rho)\varphi - \log\left(1 + e^{\varphi}\right) + \alpha\varphi_x \log \alpha\varphi_x - (\alpha\varphi_x - 1)\log(\alpha\varphi_x - 1) \Big\}\, dx - A_\alpha$$

- $h(a) = a\log a + (1-a)\log(1-a)$, A_α a constant

$$\frac{\alpha\varphi_{xx}}{\varphi_x(\alpha\varphi_x - 1)} + \frac{1}{1+e^{\varphi}} = \rho$$

$$\varphi(0) = \varphi_0 \,, \quad \varphi(1) = \varphi_1 \,, \quad \alpha\varphi_x > 1$$

Dynamical approach

- Approach does not depend on the model

$$\pi_t^N(\cdot) \;=\; \frac{1}{N}\sum_{x=1}^{N-1} \eta_{tN^2}(x)\,\delta(\cdot - x/N)$$

$$\pi_0^N \;\sim\; \rho$$

$$\mathbb{P}_\rho^N\left[\pi_t^N \;\sim\; u(t,\cdot)\right] \;\approx\; 1 \qquad \begin{cases} u_t \;+\; f(u)_x \;=\; \alpha u_{xx} \\ u(0,\cdot) = \rho(\cdot) \\ u(t,0) = \rho_0\,, \quad \rho(t,1) = \rho_1\,. \end{cases}$$

Dynamical large deviations

- Fix trajectory $u(t,\cdot)$, $t \geq 0$. Let $\rho(\cdot) = u(0,\cdot)$.

$$\mathbb{P}_\rho\{\pi^N \sim u \text{ in } [0,T]\} \approx \exp\{-N I_{[0,T]}(u)\}$$

$$I_{[0,T]}(u) \;=\; \alpha \int_0^T dt \int_0^1 \sigma(u)\,[H_x]^2\,dx$$

- $\sigma(a) = a(1-a)$ mobility

$$\begin{cases} u_t + f(u)_x - \alpha u_{xx} = -2\alpha\big[\sigma(u) H_x\big]_x \\ H(t,0) \;=\; H(t,1) \;=\; 0 \end{cases}$$

Quasi-potential

- $I_{[0,T]}(u)$: cost of observing u.
- Cost of connecting $\overline{\rho}_\alpha$ to ρ in $[-T,0]$:

$$\inf_{\substack{u(-T,\cdot)=\overline{\rho}_\alpha \\ u(0,\cdot)=\rho}} I_{[-T,0]}(u)$$

- Quasi-potential:

$$V(\rho) \;=\; \inf_{T>0} \; \inf_{\substack{u(-T,\cdot)=\overline{\rho}_\alpha \\ u(0,\cdot)=\rho}} I_{[-T,0]}(u)$$

Nonequilibrium free energy

Theorem 1 **Bodineau-Giacomin (97), Farfan (09)**
If $\overline{\rho}_\alpha$ is a global attractor, $S = V$.

$$S(\rho) \;=\; \inf_{T>0} \inf_{\substack{u(-T,\cdot)=\overline{\rho}_\alpha \\ u(0,\cdot)=\rho}} I_{[-T,0]}(u)$$

Remark: Any dimension, model.

Questions:

- Optimal trajectory.
- DLS formula for S, DLS equation
- $\alpha < \alpha_c$

Action functional

$$I_{[0,T]}(u) \;=\; \alpha \int_0^T dt \int_0^1 \sigma(u)\,[H_x]^2\,dx$$

$$\begin{cases} u_t + f(u)_x - \alpha u_{xx} = -2\alpha\big[\sigma(u) H_x\big]_x \\ H(t,0) \;=\; H(t,1) \;=\; 0 \end{cases}$$

$$I_{[0,T]}(u) \;=\; \frac{1}{4\alpha} \int_0^T dt \int_0^1 \frac{1}{\sigma(u)} \Big\{ \nabla^{-1} \big[u_t + f(u)_x - \alpha u_{xx} \big] \Big\}^2\,dx$$

$$I_{[0,T]}(u) \;=\; \int_0^T \mathcal{L}(u\,,\,u_t)\,dt$$

Hamiltonian formalism

$$I_{[-T,0]}(u) \;=\; \int_{-T}^{0} \mathcal{L}(u\,,\,u_t)\,dt$$

$$V(\rho) \;=\; \inf_{T>0} \inf_{\substack{u(-T,\cdot)=\overline{\rho}_\alpha \\ u(0,\cdot)=\rho}} I_{[-T,0]}(u)$$

$$\mathcal{H}(\rho, H) \;=\; \sup_{G} \big\{ \langle H, G\rangle \;-\; \mathcal{L}(\rho, G) \big\}$$

$$\mathcal{H}(\rho, H) \;=\; \alpha\langle \sigma(\rho), H_x^2\rangle \;+\; \langle \alpha\rho_{xx} - f(\rho)_x, H\rangle$$

Hamilton-Jacobi equation $\qquad \mathcal{H}\Big(\rho\,,\,\dfrac{\delta V}{\delta\rho}\Big) \;=\; 0$

Idea of the proof

$$\begin{cases} u_t = \alpha\, u_{xx} - f(u)_x - 2\alpha\,(\sigma(u)H_x)_x \\ H_t = -\alpha\, H_{xx} - f'(u)H_x - \alpha\,\sigma'(u)\, H_x^2 \end{cases}$$

- $(\overline{\rho}_\alpha, 0)$ is fixed point
- Hyperbolic: $(\rho, 0) \longrightarrow (\overline{\rho}_\alpha, 0)$
- $\mathcal{M}_s$ stable manifold, $\mathcal{M}_u$ unstable manifold

Statement

- $\mathcal{M}_u$ is a graph
- $\mathcal{M}_u$ is Lagrangian:
- Let $r : [0,1] \to \mathcal{M}_u$ such that $r(0) = r(1)$
- Let $r(t) = (\rho(t), H(t))$.

Then , $$\int_0^1 \langle \rho_t, H \rangle \, dt \; = \; 0 \; .$$

Claim: $$V(\rho) \; = \; C_0 \; + \; \int_\Gamma \langle \rho_t, H \rangle \, dt$$

Sketch of the proof

$$I_{(-\infty,0]}(v) \;=\; \inf_{\substack{u(-\infty,\cdot)=\overline{\rho}_\alpha \\ u(0,\cdot)=\rho}} I_{(-\infty,0]}(u) \;=\; \inf_{T>0} \; \inf_{\substack{u(-T,\cdot)=\overline{\rho}_\alpha \\ u(0,\cdot)=\rho}} I_{[-T,0]}(u)$$

$$I_{(-\infty,0]}(v) \;=\; \int_{-\infty}^{0} \mathcal{L}(v\,,\, v_t)\, dt$$

- v satisfies the Euler-Lagrange equation:

$$\frac{d}{dt}\frac{\delta\mathcal{L}}{\delta v_t}(v, v_t) \;=\; \frac{\delta\mathcal{L}}{\delta v}(v, v_t)$$

- Let $H = H(v, v_t) = (\delta\mathcal{L}/\delta v_t)(v, v_t)$
- (v, H) follows the Hamiltonian flow

Sketch of the proof

- $v(t) \longrightarrow \overline{\rho}_\alpha$ as $t \downarrow -\infty$
- Assume that $(v, H) \in \mathcal{M}_u$: $H(t) \longrightarrow 0$ as $t \downarrow -\infty$

$$
\begin{aligned}
I_{(-\infty,0]}(v) &= \int_{-\infty}^{0} \mathcal{L}(v\,,\,v_t)\,dt \\
&= \int_{-\infty}^{0} \Big\{ \langle v_t, H\rangle \;-\; \mathcal{H}(v\,,\,H) \Big\}\,dt \\
&= \int_{-\infty}^{0} \langle v_t, H\rangle\,dt \\
&= V(\rho) \;-\; V(\overline{\rho}_\alpha)
\end{aligned}
$$

DLS equation $\alpha > \alpha_c$

$$\begin{cases} u_t = \alpha\, u_{xx} - f(u)_x - 2\alpha\,(\sigma(u) H_x)_x \\ H_t = -\alpha\, H_{xx} - f'(u) H_x - \alpha\, \sigma'(u)\, H_x^2 \end{cases}$$

- Fix ρ. Let φ be given by

$$\frac{\alpha \varphi_{xx}}{\varphi_x(\alpha\varphi_x - 1)} + \frac{1}{1+e^{\varphi}} = \rho$$

$$H = \log \frac{\rho}{1-\rho} - \varphi \quad \Longrightarrow \quad (\rho, H) \in \mathcal{M}_u$$

- $\Gamma_u = \{(\rho, H)\} \subset \mathcal{M}_u$
- Γ_u is a graph by definition
- Γ_u is Lagrangian

Proof that $(\rho, H) \in \mathcal{M}_u$

- Need $(u(t), H(t)) \longrightarrow (\overline{\rho}_\alpha, 0)$ as $t \downarrow -\infty$

Equation of motion:
$$\begin{cases} u_t = \alpha\, u_{xx} - f(u)_x - 2\alpha\, (\sigma(u) H_x)_x \\ H_t = -\alpha\, H_{xx} - f'(u) H_x - \alpha\, \sigma'(u)\, H_x^2 \end{cases}$$

Time reversed:
$$\begin{cases} w_t = -\alpha\, w_{xx} + f(w)_x + 2\alpha\, (\sigma(w) J_x)_x \\ J_t = \alpha\, J_{xx} + f'(w) J_x + \alpha\, \sigma'(w)\, J_x^2 \end{cases} \quad (1)$$

Alternative formulation:
$$\begin{cases} w_t = -\alpha\, w_{xx} + f(w)_x + 2\alpha\, (\sigma(w) J_x)_x \\ J = \log \dfrac{w}{1-w} - \varphi \end{cases}$$

Miracle

$$\text{Let} \quad F = \frac{e^\varphi}{1+e^\varphi}\,. \quad \text{Then} \quad F_t \;=\; \alpha\, F_{xx} \;-\; f(F)_x$$

- Fix ρ. Solve DLS to get φ.
- Let $F = e^\varphi/(1+e^\varphi)$ and let F evolve according to the Hyd. Eq.
- Let $\varphi(t) = \log[F(t)/1 - F(t)]$. Set

$$w(t) \;=\; \frac{\alpha\varphi_{xx}(t)}{\varphi_x(t)(\alpha\varphi_x(t)-1)} + \frac{1}{1+e^{\varphi(t)}} \quad J(t) \;=\; \log\frac{w(t)}{1-w(t)} \;-\; \varphi(t)$$

Then,

- $(w(t), J(t))$ evolves according the backward equation (1)
- $w(t)$ is the hydrodynamic path of the adjoint process

Hamilton-Jacobi equation

$$\alpha \left\langle \sigma(\rho)\,,\, \left(\frac{\delta V}{\delta \rho}\right)_x^2 \right\rangle \;+\; \left\langle \alpha \rho_{xx} - f(\rho)_x\,,\, \frac{\delta V}{\delta \rho} \right\rangle \;=\; 0$$

- No unique solution, not easy to solve

Lemma: V is the maximal solution.

- Assume W solution
- Fix a trajectory u. $\quad u(-T) = \overline{\rho}_\alpha \quad u(0) = \rho$

Then , $\quad I_{[-T,0]}(u) \;\geq\; W(\rho) \;-\; W(\overline{\rho}_\alpha)$

Lower bound

$$I_{[-T,0]}(u) \;=\; \alpha \int_0^T dt \int_0^1 \sigma(u) \left(H_x - \left[\frac{\delta W}{\delta \rho}\right]_x \right)^2 dx$$

$$-\; \alpha \int_0^T dt \int_0^1 \sigma(u) \left[\frac{\delta W}{\delta \rho}\right]_x^2 dx \;+\; 2\alpha \int_0^T dt \int_0^1 \sigma(u)\, H_x \left[\frac{\delta W}{\delta \rho}\right]_x dx$$

H-J: $$-\; \alpha \int_0^T dt \int_0^1 \sigma(u) \left[\frac{\delta W}{\delta \rho}\right]_x^2 dx \;=\; \alpha \int_0^T dt \int_0^1 \left\{ u_{xx} + f(u)_x \right\} \frac{\delta W}{\delta \rho}\, dx$$

$$2\alpha \int_0^T dt \int_0^1 \sigma(u)\, H_x \left[\frac{\delta W}{\delta \rho}\right]_x dx \;=\; -2\alpha \int_0^T dt \int_0^1 \left[\sigma(u)\, H_x\right]_x \frac{\delta W}{\delta \rho}\, dx$$

$$I_{[-T,0]}(u) \;\geq\; \int_0^T dt \int_0^1 u_t \,\frac{\delta W}{\delta \rho}\, dx \;=\; W(u(0)) \;-\; W(u(-T))$$

$\alpha < \alpha_c$

$$\frac{\alpha\varphi_{xx}}{\varphi_x(1-\alpha\varphi_x)} = \frac{1}{1+e^{\varphi}} - \rho$$

- Exists a solution but not unique!
- $\Gamma_u = \{(\rho, H)\}$ is Lagrangian

$$W(\rho, H) = C_0 + \int_\Gamma \langle \rho_t, H \rangle \, dt$$

- $\Gamma_u \subset \mathcal{M}_u$
- Similar proof, but H may be discontinuous. Phase transition

$$V(\rho) = \inf \left\{ W(\rho, H) : (\rho, H) \in \Gamma_u \right\}$$