Time ordering and counting statistics

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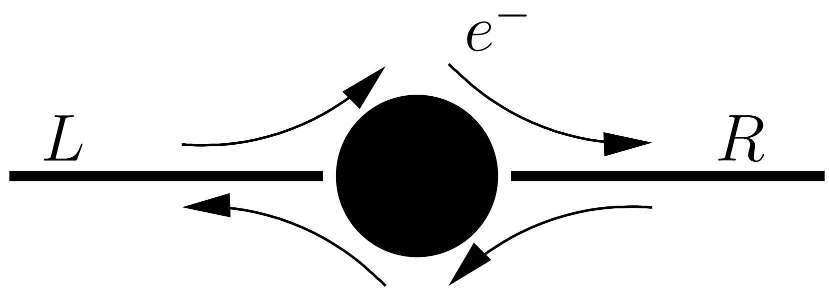
joint work with G.M. Graf and G. Lesovik

Charge transport



Typical questions: charge $\langle \mathcal{Q} \rangle$, variance $\langle\langle \mathcal{Q}^2 \rangle\rangle = \langle \mathcal{Q}^2 \rangle - \langle \mathcal{Q} \rangle^2$.

- Spatially compact device
- Consider independent fermions in the wires
- Main focus on integrated quantities over a finite time interval;
  $\rightsquigarrow$ mean current $\langle \mathcal{Q} \rangle / t$ and noise $\langle\langle \mathcal{Q}^2 \rangle\rangle / t$
- The leads are one-dimensional thermodynamic reservoirs

More generally: full statistics, *i.e.* $\langle \mathcal{Q}^k \rangle$ and $\langle\langle \mathcal{Q}^k \rangle\rangle$ for all k.

Different approaches

- Generating function of transferred charge

$$\chi(\lambda) = \sum_{n \in \mathbb{Z}} p_n \mathrm{e}^{\mathrm{i}\lambda n} = \langle \exp(\mathrm{i}\lambda \mathcal{Q}) \rangle \,;$$

  p_n is the probability for n electrons to be transported
  Well understood [Levitov-Lesovik, Avron-B-Graf-Klich]

- Integrated current correlators

$$\langle \mathcal{Q}^k \rangle \overset{?}{\sim} \int_0^t d^l t \, \langle I(t_1) \cdots I(t_l) \rangle \,;$$

  $I(t)$ is the current. Raises the question of time ordering.
  $\rightsquigarrow$ Controversies

Matthews' T^* ordering

The answer is given by the following T^* ordering of current operators

$$\langle \mathcal{Q}^k \rangle = \int_0^t d^k t \, \langle \mathrm{T}^* \left(I(t_1) \cdots I(t_k) \right) \rangle \,,$$

where

$$\mathrm{T}^* \left(I(t_1) \cdots I(t_k) \right) = \frac{\partial}{\partial t_k} \cdots \frac{\partial}{\partial t_1} \mathrm{T} \left(Q(t_1) \cdots Q(t_k) \right) \,,$$

and T is the standard time ordering for charge operators $Q(t)$.

Remark. T^* was introduced by Matthews in '49 for QFT correlators $\rightsquigarrow$ Causal perturbation theory.

Contact terms

The derivatives are outside of T,

$$\frac{\partial}{\partial t}\,\mathrm{T}\,A(t)B(s) = \mathrm{T}\,\dot{A}(t)B(s) + \delta(t-s)[A(t), B(t)]\,.$$

Hence the general structure of T^*

$$\mathrm{T}^*\left(I(t_1)\cdots I(t_k)\right) = \sum_{P\in\mathcal{P}_k}\mathrm{T}\left(\prod_{C\in P}\mathrm{ad}_Q^{n(C)-1}(I)(t_C)\delta_C\right),$$

where the sum runs over all partitions P of $\{1\ \ldots, k\}$ into nonempty disjoint subsets C of cardinality $n(C)$. δ_C represents a product of δ-functions collapsing the times t_i, $(i \in C)$ to a single $t_C = t_i$.

Remarks

- For $k = 2$,

$$\mathrm{T}^* \left(I(t_1)I(t_2)\right) = \mathrm{T}\left(I(t_1)I(t_2)\right) + \delta(t_1 - t_2)[Q(t_1), I(t_1)]\,.$$

- The partition into single-element clusters $(1)(2)\dots(k)$ contributes precisely

$$\mathrm{T}\left(I(t_1)\cdots I(t_k)\right)\,.$$

- All other terms have a singular support on some time coincidences due to δ_C, $\rightsquigarrow$ 'contact terms'.
- The contact terms vanish if $[Q, I] = 0$.

A first derivation

From basic measurement axioms of quantum mechanics,

$$\chi(\lambda) = \left\langle U^* \mathrm{e}^{\mathrm{i}\lambda Q} U \mathrm{e}^{-\mathrm{i}\lambda Q} \right\rangle = \left\langle \mathrm{e}^{\mathrm{i}\lambda Q(t)} \mathrm{e}^{-\mathrm{i}\lambda Q} \right\rangle = \left\langle \mathrm{T}\, \mathrm{e}^{\mathrm{i}\lambda (Q(t) - Q)} \right\rangle .$$

and hence

$$\langle \mathcal{Q}^k \rangle = \left\langle \mathrm{T} \left((Q(t_1) - Q) \cdots (Q(t_k) - Q) \right) \right\rangle \Big|_{t_1 = \ldots = t_k = t} .$$

By repeated use of the fundamental theorem of calculus,

$$\langle \mathcal{Q}^k \rangle = \int_0^t d^k t \, \frac{\partial}{\partial t_k} \cdots \frac{\partial}{\partial t_1} \left\langle \mathrm{T} \left((Q(t_1) - Q) \cdots (Q(t_k) - Q) \right) \right\rangle ,$$

and the terms in $-Q$ do not contribute to the derivative.

More involved...

Let $U = \exp(-\mathrm{i}Ht)$, then

$$\chi(\lambda) = \left\langle U^* \left(\mathrm{e}^{\mathrm{i}\lambda Q} U \mathrm{e}^{-\mathrm{i}\lambda Q}\right)\right\rangle = \left\langle \mathrm{e}^{\mathrm{i}Ht} \mathrm{e}^{-\mathrm{i}H(\lambda)t} \right\rangle \;,$$

where

$$\begin{aligned} H(\lambda) &= \mathrm{e}^{\mathrm{i}\lambda Q} H \mathrm{e}^{-\mathrm{i}\lambda Q} \\ &= H + \mathrm{i}\lambda[Q, H] - \lambda^2/2[Q,[Q,H]] - \mathrm{i}\lambda^3/6[Q,[Q,[Q,H]]] + \ldots \\ &= H - \lambda I - \mathrm{i}\lambda^2/2[Q, I] + \lambda^3/6[Q,[Q,I]] + \ldots \end{aligned}$$

$\chi(\lambda)$ is the expectation of the propagator in the interaction picture for H.
$\rightsquigarrow$ Dyson expansion yields the contact terms as well.

Second quantization

- The pump is an open system with fermionic reservoirs. We consider gauge invariant quasi-free states

$$\langle a^*(f)a(g)\rangle_\rho = (g, \rho f)\,,$$

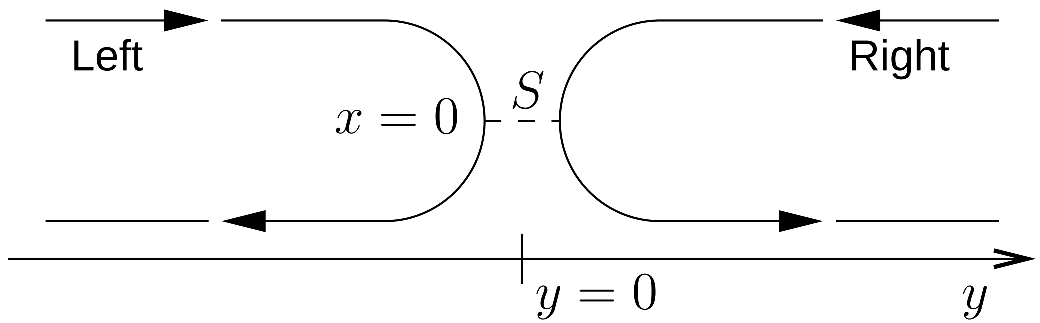
and use the GNS representation $(\mathcal{H}_\rho, \pi_\rho, \omega_\rho)$.

- Second quantization: $A \mapsto \widehat{A}$, s.t. $\langle\widehat{A}\rangle_\rho = 0$.
- There appear Schwinger terms, namely

$$[\widehat{A}, \widehat{B}] = \widehat{[A, B]} + \big(\operatorname{tr}\big(\rho A \rho' B \rho\big) - \operatorname{tr}\big(\rho' A \rho B \rho'\big)\big) \cdot 1$$

(recall, $\rho = 0$ is the Fock representation)
Contact terms come in the guise of Schwinger terms.

Instantaneous scattering



Chiral leads, $\mathcal{H} = L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ and potential drop V.
Constant, instantaneous scattering over $[0, t]$, given by the scattering matrix S, superimposed on otherwise free motion:

$$(U(t)\psi)(x) = \psi(x - t) + (S - 1)\,\theta(0 < x < t)\psi(x - t)\,,$$

Importantly,

$$[Q(t), I(t)] = [Q, S^*QS]\,\delta(x + t) \neq 0\,.$$

T^* in action

We use the T^* ordering for the third cumulant as $t \to \infty$,

$$\begin{aligned}\langle\!\langle \mathcal{Q}^3(t) \rangle\!\rangle &= \int_0^t d^3t \, \langle\!\langle \mathrm{T}\, \widehat{I}_1 \widehat{I}_2 \widehat{I}_3 \rangle\!\rangle + 3 \int_0^t d^2t \, \langle\!\langle \mathrm{T}\, \widehat{I}_1 [\widehat{Q}_2, \widehat{I}_2] \rangle\!\rangle \\ &\quad + \int_0^t dt_1 \, \langle\!\langle [\widehat{Q}_1, [\widehat{Q}_1, \widehat{I}_1]] \rangle\!\rangle \\ &= -\frac{Vt}{2\pi} \, 2T^2(1-T) + 0 + \frac{Vt}{2\pi} T(1-T) \\ &= \frac{Vt}{2\pi} \cdot T(1-T)(1-2T)\,.\end{aligned}$$

Binomial statistics in the long time limit, as predicted from an explicit computation of the generating function of transferred charge.

Evolution, charge and current

- Time evolution $(t > 0)$: Instantaneous scattering at $x = 0$

$$(U(t)\psi)(x) = \psi(x - t) + (S - 1)\,\theta(0 < x < t)\psi(x - t)\,,$$

- The charge

$$Q(t) = Q(\theta(-x - t) + \theta(x)) + S^*QS\,\theta(x + t)\theta(-x)\,,$$

- The current

$$I(t) = \frac{dQ(t)}{dt} = (S^*QS - Q)\,\delta(x + t) = I_+(t) + I_-(t)\,.$$

- Importantly,

$$[Q(t), I(t)] = [Q, S^*QS]\,\delta(x + t) \neq 0\,.$$

Determinants

- First, [Lesovik-Levitov, Avron-B.-Graf-Klich]

$$\left\langle \widehat{U}^* \mathrm{e}^{\mathrm{i}\lambda \widehat{Q}} \widehat{U} \mathrm{e}^{-\mathrm{i}\lambda \widehat{Q}} \right\rangle_\rho = \det \left(\mathrm{e}^{-\mathrm{i}\lambda \rho_U Q_U} \rho' \mathrm{e}^{\mathrm{i}\lambda \rho Q} + \mathrm{e}^{\mathrm{i}\lambda \rho'_U Q_U} \rho \mathrm{e}^{-\mathrm{i}\lambda \rho' Q} \right)$$

  where $\widehat{O}$ denotes the second quantized operators, $\rho' = 1 - \rho$ and $O_U = U^* O U$.
- Consider now $\rho_i(p) = \theta(\mu_i - p)$: Fermi seas at zero temperature, with 'voltage bias' $V = \mu_L - \mu_R$
- The determinant is a well-defined Fredholm determinant
- The operator in the $\det$ is of the form (Wiener-Hopf)

$$\psi(x) \longmapsto \psi(x) + \int_{-t}^{0} dy\, A(x-y)\psi(y)\,,$$

Binomial statistics

- Its determinant can be computed using Kac-Akhiezer

$$\frac{1}{t}\log\det(1+A) \xrightarrow{t\to\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty}\log(1+(\mathcal{F}A)(k))dk\,,$$

  where $(\mathcal{F}A)(k)$ is the Fourier transform of the function $A(x)$.

- Hence,

$$\chi_\rho(\lambda) = \left((1-T)+\mathrm{e}^{\mathrm{i}\lambda}T\right)^{(tV)/(2\pi)}(1+o(t))\,,\quad (t\to\infty)\,.$$

- Represents a binomial distribution of probability T (the transmission probability) and number of trials $N \propto tV$.
- In particular, $\langle\langle Q^3(t)\rangle\rangle = \frac{Vt}{2\pi}T(1-T)(1-2T)\,.$

In/out currents

- In the scattering approach: $\widetilde{\mathrm{T}}$ ordering [Beenakker] Places incoming currents $I_-(t)$ at the right of outgoing $I_+(s)$, independently of their time indices.
- In this model, $\mathrm{T}^* \big(I(t_1)\cdots I(t_n)\big) = \widetilde{\mathrm{T}} \big(I(t_1)\cdots I(t_n)\big)$.
- Moreover, $\mathrm{T}^* \big(\widehat{I}(t_1)\cdots \widehat{I}(t_n)\big) = \widetilde{\mathrm{T}} \big(\widehat{I}(t_1)\cdots \widehat{I}(t_n)\big)$ *i.e.* Equivalence of scattering approach and counting statistics
- Statement about $\widehat{I}$ nontrivial because of Schwinger terms:

$$[\widehat{Q}, \widehat{I}] = \widehat{[Q, I]} + \big(\mathrm{tr}\big(\rho Q \rho' I \rho\big) - \mathrm{tr}\big(\rho' Q \rho I \rho'\big)\big) \cdot 1$$

related to infinite depth of Fermi sea.

Measurement procedure

$\mathcal{H}$, Hilbert space of pure states; ρ, a density matrix,
$Q = \sum_i n_i P_i$, an observable. Two steps protocol:

(i) Measure Q initially, $\rho \rightsquigarrow \sum_i P_i \rho P_i$

(ii) Let the evolution U act for time t

(iii) Measure Q finally, $\sum_i U P_i \rho P_i U^* \rightsquigarrow \sum_{i,j} P_j U P_i \rho P_i U^* P_j$

The probability of the history (n_i, n_j) is $\mathrm{tr}(P_j U P_i \rho P_i U^* P_j)$, i.e.

$$\chi_\rho(\lambda) = \sum_{i,j} \mathrm{tr}(U^* P_j U P_i \rho P_i) \mathrm{e}^{\mathrm{i}\lambda(n_j - n_i)} = \sum_i \mathrm{tr}(U^* \mathrm{e}^{\mathrm{i}\lambda Q} U P_i \rho P_i) \mathrm{e}^{-\mathrm{i}\lambda n_i}$$

If $[Q, \rho] = 0$,

$$\chi_\rho(\lambda) = \mathrm{tr}(U^* \mathrm{e}^{\mathrm{i}\lambda Q} U \mathrm{e}^{-\mathrm{i}\lambda Q} \rho) = \left\langle U^* \mathrm{e}^{\mathrm{i}\lambda Q} U \mathrm{e}^{-\mathrm{i}\lambda Q} \right\rangle_\rho \ .$$

Dyson expansion

Write the propagator in a Dyson expansion,

$$\mathrm{e}^{\mathrm{i}Ht}\mathrm{e}^{-\mathrm{i}H(\lambda)t} = \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} \int_0^t d^n t \, \mathrm{T}\left(W(t_1)\cdots W(t_n)\right)$$

with $W(t) = \exp(\mathrm{i}Ht)W\exp(-\mathrm{i}Ht)$.
Terms of order λ^k are found for all $n \leq k$, and $\langle \mathcal{Q}^k \rangle$ equals

$$\sum_{\substack{n,(j_1,\ldots j_n) \\ \sum_i j_i = k}} \frac{k!}{n! j_1! \cdots j_n!} \int_0^t d^n t \, \langle \mathrm{T}\left(\mathrm{ad}_Q^{(j_1-1)}(I)(t_1)\cdots \mathrm{ad}_Q^{(j_n-1)}(I)(t_n)\right)\rangle$$

where $\mathrm{ad}_A^0(B) = B$ and $\mathrm{ad}_A^j(B) = [A, \mathrm{ad}_A^{j-1}(B)]$ for $j > 0$.