Mathematical Foundation of the Brockett-Wegner Flow

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Brockett's Idea (1991)

- Let $H = H^*, A = A^* \in \mathcal{B}[\mathbb{C}^N]$,
- H is $\underline{A\text{-diagonal}}$ $:\Leftrightarrow \quad [H, A] = 0$.
- For $A = \mathrm{diag}(\alpha_1, \ldots, \alpha_N)$ with $\alpha_j < \alpha_{j+1}$:
  H is A-diagonal $\Leftrightarrow H = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$.
- Study the time-dependent Schrödinger Eq.

$$\forall t > 0 : \quad \partial_t H_t \; := \; i[H_t, G_t], \qquad H_0 \; := \; H,$$

  where $G = G^* \in C^1\big(\mathbb{R}_0^+; \mathcal{B}[\mathbb{C}^N]\big)$ is chosen later.
- Introduce Lyapunov function

$$f_t \; := \; \frac{1}{2}\mathrm{Tr}\big\{\big(H_t - A\big)^2\big\} \; := \; \frac{1}{2}\mathrm{Tr}\big\{H^2 + A^2 - 2H_t A\big\} \; \geq \; 0.$$

  Then

$$\dot{f}_t \; := \; -\mathrm{Tr}\big\{\dot{H}_t \, A\big\} \; = \; -\mathrm{Tr}\big\{i[H_t, G_t]A\big\} \; = \; -\mathrm{Tr}\big\{i[A, H_t]G_t\big\}.$$

- Brockett's key observation:

$$G_t = G_t^* := i[A, H_t] \qquad \Rightarrow \qquad \dot{f}_t = -\mathrm{Tr}\big\{G_t^2\big\} \leq 0.$$

Fund. thm. of calc. implies $-\dot{f} \in L^1(\mathbb{R}^+; \mathbb{R}_0^+)$, i.e., $\mathrm{Tr}\big\{[iA, H_t]^2\big\} \; \to \; 0$.

• If convergent then

$$H_\infty \;=\; \lim_{t\to\infty} H_t \;=\; \lim_{t\to\infty} \big\{ U_{t,0} H U_{t,0}^* \big\}$$

which is unitarily equivalent to H and A-diagonal. $(U_{t,s})_{t\geq s\geq 0} \subseteq \mathrm{U}(N)$ is the cocycle of unitary matrices generated by iG_t.

• Similar idea of Wegner (1994): Choosing $G_t := i\big[H_t^{\mathrm{diag}}\,,\; H_t\big]$, he shows that $\mathrm{Tr}\big\{G_t^2\big\}$ is monotonically decreasing.

• Brockett proposed method for diagonalization of matrices and to solve linear programs.

• Wegner applied his method to many-particle Hamiltonians; Kehrein and Mielke [(IR-singular) spin-boson model, 1994-1996], Stein (Hubbard model, 1998), [see Kehrein 2006].

• As a (nonlinear) evolution equation,

$$\forall t > 0: \quad \partial_t H_t \;:=\; \big[H_t, [H_t, A]\big], \qquad H_0 \;:=\; H,$$

• Finiteness and cyclicity of the trace on $\mathbb{C}^N$ was used at several places.

Global Existence for Bounded Operators

Thm 1: Let $(\mathcal{A}, \| \cdot \|_\mathcal{A})$ be a Banach subalgebra of the Banach algebra $\mathcal{B}[\mathfrak{H}] \supset \mathcal{A}$ of bounded operators on a separable Hilbert space $\mathfrak{H}$ such that $\| \cdot \|_\mathcal{A}$ is a unitarily invariant norm. Suppose that $H_0 = H_0^*, A = A^* \in \mathcal{A}$ are two self-adjoint operators such that $A \geq 0$. Then

$$\forall t > 0 : \quad \partial_t H_t \ := \ \big[H_t, [H_t, A]\big], \qquad H_0 \ := \ H, \tag{1}$$

has a unique solution $H_{(\cdot)} \in C^\infty(\mathbb{R}_0^+; \mathcal{A})$, and H_t is unitarily equivalent to H_0, for all $t > 0$.

<u>Proof:</u>
The right side of (1) is locally Lipschitz continous.

$\Rightarrow$ Local existence $H_{(\cdot)} \in C^\infty([0, T]; \mathcal{A})$, for some $T > 0$.

$\|H_T\|_\mathcal{A} = \|H_0\|_\mathcal{A}$ because of unitary equivalence.

$\Rightarrow$ Global existence by iteration.

Local Existence for Unbounded Operators

Let $\mathfrak{D} \subset \mathfrak{H}$ be a dense domain, $X = \mathcal{B}[\mathfrak{H}]$ and $Y = \mathcal{B}[\mathfrak{D}]$, $H_0 = H_0^*$, $A = A^* \in \mathcal{B}[\mathfrak{D}; \mathfrak{H}]$ such that $A \geq 1$. Further assume $R_n(H_0) \in X \cap Y$, where

$$R_0(H) := A^{-1}, \quad R_1(H) := [H, A^{-1}], \quad R_2(H) := \big[H, [H, A^{-1}]\big], \ldots$$

Thm 2: Suppose that

$$\sum_{n=0}^{\infty} \frac{e^{\rho n}}{n!} \big(\|R_n(H_0)\|_X + \|R_n(H_0)\|_Y \big) \ \leq \ e^{\eta}\,, \tag{2}$$

for some $\rho, \eta \in \mathbb{R}$. Then

$$\forall t > 0: \quad \partial_t H_t \ := \ \big[H_t, [H_t, A^{-1}]\big], \qquad H_0 \ := \ H, \tag{3}$$

has a self-adjoint solution $H_{(\cdot)} \in C^\infty\big([0, T_*); \mathcal{B}[\mathfrak{D}, \mathfrak{H}]\big)$, where $T_* := \frac{1}{8} e^{\rho - \eta - 1}$, and there exists a smooth family $U_{(\cdot)} \in C^\infty\big([0, T_*); X \cap Y\big)$ of unitary tranformations preserving the domain $\mathfrak{D}$ of H_t such that $H_t = U_t H_0 U_t^*$ is unitarily equivalent to H_0, for all $t \in [0, T_*)$. It obeys

$$\sum_{n=0}^{\infty} \frac{e^{\rho n}}{n!} \max_{t \in [0,T]} \big\{ e^{-nt/T_*} \big(\|R_n(H_t)\|_X + \|R_n(H_t)\|_Y \big) \big\} \ \leq \ 2\, e^{\eta} \tag{4}$$

and is the only solution with this property.

- Proof is inspired by Caps 2002 and uses Nash-Moser type estimates.

Convergence on Hilbert-Schmidt Operators

Thm 3: Suppose $H_0 := H = H^*, A = A^* \in \mathcal{L}^2[\mathfrak{H}]$ are two self-adjoint Hilbert-Schmidt operators on a separable Hilbert space $\mathfrak{H}$ such that $A > 0$ has full rank, and let $H_{(\cdot)} \in C^\infty\big(\mathbb{R}_0^+; \mathcal{L}^2[\mathfrak{H}]\big)$ be the unique solution of (1). Then

$$H_\infty := \lim_{t\to\infty} H_t$$

converges strongly, H_∞ is A-diagonal, and H_∞ is unitarily equivalent to H_0.

• Note that, while $H_t = U_{t,0} H_0 U_{t,0}^*$, there is no claim about the convergence of $U_{t,0}$, as $t \to \infty$.

Diagonalization of Quadratic Operators

- For a_k^*, a_ℓ obeying the CCR and H_t of the form

$$H_t \;=\; \sum_{k,\ell} \Big\{ \Omega_t(k,\ell)\, a_k^*\, a_\ell + B_t(k,\ell)\, a_k^*\, a_\ell^* + \overline{B_t(k,\ell)}\, a_k\, a_\ell \Big\}$$

- Choosing $A := \sum_k a_k^*\, a_k$, we have:

  H_t is A-diagonal $\Leftrightarrow$ H_t is in normal form (Berezin)

- Eq. (1) is equivalent to system

$$\partial_t \Omega_t \;=\; -16\, B_t\, B_t^*, \qquad \partial_t B_t \;=\; -2\big(\Omega_t\, B_t + B_t\, \Omega_t^T\big). \tag{5}$$

- Exist. + uniq. + conv. under the assumption that, for some $\varepsilon > 0$

$$\big(\Omega_0^{-3/2} + \mathbf{1}\big) B_0 \in \mathcal{L}^2[\mathfrak{H}], \quad 8 B_0 \Omega_0^T B_0^* \leq \Omega_0, \quad (4+\varepsilon) B_0 \Omega_0^T B_0^* \leq \mathbf{1}.$$

- Weaker than standard assumption $\Omega_0 \geq \mu \cdot \mathbf{1}$ ($\Rightarrow \exists$ gap),
- $4 B_0 \Omega_0^T B_0^* \leq \mathbf{1}$ is necessary for H_t to be bounded below.