A SUSY model of quantum transport in a random environment

Joint work with:
M. Disertori and M. Zirnbauer

June 9, 2009

Motivation

To study time evolution of *quantum* particle in a random environment using **SUSY** Statistical Mechanics.

> Averages of Greens functions can be expressed *exactly* as correlations in a **SUSY** Statistical Mechanics Model.
> (F.Wegner, K. Efetov)

Advantages: Disorder is integrated out, Saddle point analysis can be applied, Symmetries governing universality emerge. Extended states closely connected to symmetry breaking.

Disadvantages: Spin is a 4x4 supermatrix, lack of positivity, Hyperbolic $SU(1,1|2)$ Symmetry.

Main Result

Theorem: "Diffusion" for a Simpler 3D SUSY Hyperbolic Sigma model.

Spin has 4 components: $(t_j, s_j, \psi_j, \overline{\psi_j})$ s, t real, $j \in \mathbb{Z}^3$

Model *expected* to reflect Localization-Diffusion phenomena.

Equivalent to random walk in a *correlated* random environment.

Random environment is given by a **real**, **non convex**, **nonlocal** action obtained after integrating out Grassmann $\psi_j, \overline{\psi_j}$.

Ingredients of Proof:

Need to estimate fluctuations of the environment.

Key is to use **SUSY** Ward Identities. Eg. $Z(\beta_{ij}) \equiv 1$

Estimates on **nonuniformly** elliptic equations.

Control fluctuations on scale ℓ, by induction.

No cluster expansion.

Relation to Edge Reinforced Random Walk?

This *history* dependent random walk favors edges it has visited in the past.

It is also equivalent to a random walk
in a *correlated* random environment (Diaconis).

This environment looks very much like ours!

Localization in 1D , Merkl and Rolles
Use a Mermin-Wagner type argument to estimate fluctuations of the environment - good lower lower bounds on localization length. Partial results in 2D.

Quantum Green's Function

Let **H** denote a *random* band matrix or Schrödinger operator.

$$G_\epsilon(E; x, y) \equiv (\mathbf{H} - E + i\epsilon)^{-1}(x, y).$$

Q-Diffusion Conjecture in 3D:

$$< |G_\epsilon(E; x, y)|^2 > \cong \frac{\rho(E)}{-D\Delta + \epsilon}(x, y) \approx C\,(|x - y| + 1)^{-1}$$

$\rho(E) \;=\; Im < G_\epsilon(E; 0, 0) >=$ density of states.

$D = D(E) > 0$ is diffusion constant.

Localization :

$$< |G_\epsilon(E; x, y)|^2 > \cong\; \epsilon^{-1}\, e^{-|x-y|/\ell}$$

SUSY Hyperbolic Sigma model

Sigma constraint: $-z_j^2 + x_j^2 + y_j^2 + 2\overline{\psi_j}\psi_j = -1.$

Action $\mathbf{A}$ in *horospherical* coordinates:

$$t_j,\ s_j \in \mathbb{R},\ \ \textit{and}\ \ \overline{\psi}_j, \psi_j\ \ \textit{Grassmann}$$

$$\mathbf{A}_\epsilon(\mathbf{t}, \mathbf{s}, \psi) \equiv \boldsymbol{\Sigma}_{\mathbf{j}\sim\mathbf{j}'}(\mathbf{cosh}(\mathbf{t_j} - \mathbf{t_{j'}}) - \mathbf{1}) + \frac{\epsilon}{\beta}\sum_{\mathbf{j}} \mathbf{cosh}\ \mathbf{t_j}$$

$$+ \sum_{\mathbf{j}\sim\mathbf{j}'} \mathbf{e}^{(\mathbf{t_j}+\mathbf{t_{j'}})}\Big[\frac{\mathbf{1}}{\mathbf{2}}(\mathbf{s_j} - \mathbf{s_{j'}})^2 + (\overline{\psi}_\mathbf{j} - \overline{\psi}_{\mathbf{j'}})(\psi_\mathbf{j} - \psi_{\mathbf{j'}})\Big]$$

$$Z = \int e^{-\beta A_\epsilon(t,s,\psi)} \prod_j e^{-t_j} dt_j ds_j d\psi_j d\overline{\psi_j} \equiv 1$$

Elliptic generator **D** with random conductances

For $t_j \in \mathbb{R}$, define $\mathbf{D(t)}$ via the quadratic form:

$$(f, \mathbf{D}_\beta(t)\, f) = \beta \sum_{j \sim j'} e^{t_j + t_{j'}} \, (f(j) - f(j'))^2.$$

D is the generator of *random walk in the random environment* of the t field. **D** is **not** uniformly elliptic.

If $e^{(t_j + t_{j'})} \sim 1$ then there is **diffusion**.

However, if $< e^{t_j/2} >_{\beta,\epsilon} << 1$, then *conductance goes to zero* and **localization** occurs. Known in 1D and expected in 2D

Relation to Quantum-Green's Function

$$< |G_\epsilon(E; x, y)|^2 >_{RM} \cong < s_x e^{t_x} s_y e^{t_y} >_{SUSY} (\beta, \epsilon)$$

$$= < [\beta \, \mathbf{D} \ (t) + \epsilon e^{t}]^{-1}(x, y) e^{t_x + t_y} >_{SUSY} (\beta, \epsilon)$$

$$= < [-\beta \ \Delta + V(t) + \epsilon e^{-t}]^{-1}(x, y) >_{SUSY} (\beta, \epsilon)$$

$\beta \sim$ density of states $\rho(E)$, times band width.

Main Theorem: "Diffusion" in 3D (Di-Sp-Zi):

For β **large**, local conductance $e^{t_j+t_{j'}} \sim 1$, hence :

$$< e^{t_0+t_x}[\beta \mathbf{D}(t) + \epsilon e^t]^{-1}(0,x) >_{SUSY} (\beta)$$

$$\approx (-\beta\Delta + \epsilon)^{-1}(0,x) \approx C|x|^{-1}$$

So, the evolution is **"Diffusive"**.

Conjecture: For **small** β, **localization** occurs in 3D:
$< e^{t_j/2} > \to 0$ as $\epsilon \to 0$, conductance vanishes.

Saddle point

Let $B_{\beta,\epsilon}(t)$ be effective action (after integrating out s and ψ).

$B_{\beta,\epsilon}(t)$ a unique minimum at t_s , depending on β and ϵ:

$$\epsilon e^{-t_s} = 1/\beta \qquad 1 \ \textit{Dim}$$

$$\epsilon e^{-t_s} = e^{-\beta} \qquad 2 \ \textit{Dim}$$

$$t_s \cong 0 \qquad 3 \ \textit{Dim for } \beta \textit{ large}$$

$$\epsilon e^{-t_s} \sim 1 \qquad 3 \ \textit{Dim for } \beta \textit{ small}$$

Saddle *suggests that localization occurs in 1 and 2 dimensions and diffusion occurs in 3D. Zirnbauer proved localization in 1D.*

Ward Identities are Key to Proof

To estimate the conductance, we first bound the *fluctuations* of the t field,

$$< cosh^m(t_0 - t_\ell) > (\beta, \epsilon) \leq Const.$$

by **induction** on $|\ell|$ and **Ward identities:**

Example: For $\ell \in \mathbb{Z}^d$, *let*

$$F_\ell = cosh(t_0 - t_\ell) + \; e^{t_0 + t_\ell} \Big[\frac{1}{2}(s_0 - s_\ell)^2 + (\overline{\psi}_0 - \overline{\psi}_\ell)(\psi_0 - \psi_\ell) \Big]$$

$$Then \;\; < F_\ell^m > = 1 \;\; all \; m \geq 1.$$

After integrating over the ψ and s variables

The Ward identity $< F_\ell^m >= 1$ yields:

$$< cosh^m(t_0 - t_\ell)(1 - \frac{m}{\beta} G_\ell(t)) >= 1$$

where G_ℓ is the Green's function of $D(t)$.

If $0 \le G_\ell(t) \le \mathbf{C}$, then for $\mathbf{C}m < \beta$

$$< cosh^m(t_0 - t_\ell) > \le (1 - \frac{\mathbf{C}m}{\beta})^{-1}$$

Thus t - fluctuations at scale ℓ are small for large β.

Conclusions and open Problems

Quantum Dynamics $\approx$ SUSY hyperbolic model $=$ RWRE.

In our Hyperbolic SUSY model, localization and diffusion are revealed using a simple **saddle** analysis. But **fluctuations** need to be **estimated**. We do this in 3D for large β.

Once the Ward identities are established, analysis is classical: Induction on length scales $+$ estimates on elliptic Greens functions.

Problem: Prove Localization in 2D and Phase Transition in 3D.