LIMITING DYNAMICS OF SOLITONS IN RANDOM POTENTIALS

CTS, Zurich. June 8, 2009.

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Joint work with C. Sulem.

PURPOSE

- Study the dynamics of solitons for the generalized nonlinear Schödinger equation (NLSE) in random (and time-dependent) external potentials.

- Investigate whether the analogy between solitons and *"point particles* $+ \epsilon$" over certain spatial and temporal scales holds for random potentials.

- Search for regimes with interesting limiting dynamics for the center of mass of the soliton.

Why might it be interesting?

NLSE describes very many nonlinear phenomema, ranging from BEC (GP-equation), nonlinear optics, to biology. Solitons may propagate in the field of random scatterers or inhomogeneities.

Dynamics and stability of NLSE solitons in random potentials poorly understood.

METHODS

• Advances in nonlinear PDEs in the past two decades: well-posedness of NLSE, existence and stability of solitons, modulation theory . [Kato, Ginibre, Velo, Berestycki, Lions, Strauss, Grillakis, Shatah, Weinstein]

• Progress in understanding effective motion of solitons in deterministic external potentials. [Jerrard, Bronski, Fröhlich, Yau, Tsai, Gustafson, Sigal, Jonsson, Zworski, Holmer, A-S]

• Understanding limiting dynamics of classical particles in random potentials. [Papanicolaou, Kesten, Lebowitz, Dürr, Goldstein, Ryzhik, Komorowski]

SOME HISTORY

Bronski and Jerrard, 2000: semi-classical limit of solitary wave dynamics.

Fröhlich, Tsai and Yau, 2000 - 2002: effective dynamics for Hartree equation.

Fröhlich, Gustafson, Jonsson and Sigal, 2004 - 2006: effective dynamics for generalized NLSE.

Holmer and Zworski, 2007-2008: effective dynamics in the presence of a delta potential and in the presence of slowly varying potential for the cubic NLSE in 1-dimension.

A-S, 2007-2008: effective dynamics in the presence of a time-dependent potential, and in the presence of rough nonlinear perturbations.

A-S, Fröhlich and Sigal, 2008: collision of fast solitons in an external potential.

A-S and Sulem, 2009: resonance tunneling of fast solitons through large potential barriers.

SOME RELATED RESULTS

Garnier, 1998: (formal) analysis of soliton transmission in random media (in 1-dimension).

Di Mensa, 2006: numerical study of diffusion of soliton in the presence of noise.

Erdos, Yau, Salmhofer, 2007: quantum diffusion for the linear Schrödinger equation with random potential in the weak-coupling/semi-classical limit.

Bourgain and Wang, 2008: localization for NLSE with random potential.

NLSE with random potential describes the mean-field dynamics of many-body bosons in a random potential. [A-S]

THE MODEL

NLSE

$$i\partial_t\psi(\mathbf{x},t) = (-\Delta+\lambda V_h(\mathbf{x},t;\omega))\psi(\mathbf{x},t)-f(\psi(\mathbf{x},t))$$

Probability triple $(\Omega, \mathcal{F}, \mathbb{P})$

External potential

$$V_h(\mathbf{x}, t; \omega) \equiv V(h\mathbf{x}, t; \omega), \; h \in (0, 1]$$

$$\omega \in \Omega, \quad \mathbf{x} \in \mathbb{R}^N, \quad t \in \mathbb{R}$$

$$V_h \in L^\infty(W^{1,\infty}(\mathbb{R}, C^2(\mathbb{R}^N) \cap W^{2,\infty}(\mathbb{R}^N)))$$

Nonlinearity

$$f : H^1(\mathbb{R}^N; \mathbb{C}) \to H^{-1}(\mathbb{R}^N; \mathbb{C})$$

with $f(0) = 0$, and $\overline{f(\psi)} = f(\overline{\psi})$.

Typical nonlinearities

$$f(\psi) = |\psi|^s \psi, \quad 0 < s < \frac{4}{N}.$$

Solitary wave solution

$$\eta_\sigma(\mathbf{x}, t) := e^{i(\frac{1}{2}\mathbf{v}\cdot(\mathbf{x}-\mathbf{a})+\gamma)}\eta_\mu(\mathbf{x}-\mathbf{a})$$

$$\sigma := (\mathbf{a}, \mathbf{v}, \gamma, \mu)$$

$$(-\Delta + \mu)\eta_\mu - f(\eta_\mu) = 0$$

Initial condition

Close to a soliton

$$\|\psi(t=0)-\eta_{\sigma_0}\|_{H^1}\le C_0 h$$

EFFECTIVE DYNAMICS

Theorem 1

For all $h \in (0, h_0)$ and for any fixed $\epsilon \in (0,1)$, we have that

$$\sup_{t\in[0,\overline{C}\epsilon|\log h|/\lambda h)} \mathbb{E}[\|\psi - \eta_{\sigma(t)}\|_{H^1}] \leq Ch^{1-\frac{\epsilon}{2}},$$

uniformly in $\lambda \in (h^{1-\epsilon}, 1]$, and

$$\begin{aligned}
\partial_t \mathbf{a} &= \mathbf{v} + O(h^{2-\epsilon}) \\
\partial_t \mathbf{v} &= -2\lambda \nabla V_h(\mathbf{a}, t; \omega) + O(h^{2-\epsilon}) \\
\partial_t \gamma &= \mu + \frac{1}{4}\|\mathbf{v}\|^2 - V_h(\mathbf{a}, t; \omega) + O(h^{2-\epsilon}) \\
\partial_t \mu &= O(h^{2-\epsilon}),
\end{aligned}$$

with

$$\|\mathbf{a}(0) - \mathbf{a}_0\|, \|\mathbf{v}(0) - \mathbf{v}_0\|, |\gamma(0) - \gamma_0|, |\mu(0) - \mu_0| = O(h).$$

WEAK-COUPLING/SPACE-ADIABATIC LIMIT

Suppose that $V = \overline{V}$ is time-independent. We make the scaling

$$\overline{\mathbf{a}} := h\mathbf{a}$$
$$\overline{\mathbf{v}} := \mathbf{v}$$
$$\overline{t} := ht.$$

It follows that

$$\partial_{\overline{t}}\overline{\mathbf{a}} = \overline{\mathbf{v}} + O(h^{2-\epsilon})$$
$$\partial_{\overline{t}}\overline{\mathbf{v}} = -2\lambda\nabla\overline{V}(\overline{\mathbf{a}};\omega) + O(h^{1-\epsilon})$$

with initial condition

$$\|\overline{\mathbf{a}}(0) - h\mathbf{a}_0\| = O(h^2),\ \|\overline{\mathbf{v}}(0) - \mathbf{v}_0\| = O(h).$$

Auxiliary stochastic process

$(\tilde{\mathbf{a}}(\bar{t}), \tilde{\mathbf{v}}(\bar{t}))_{\bar{t} \geq 0}$ given by

$$\partial_{\bar{t}} \tilde{\mathbf{a}} = \tilde{\mathbf{v}}$$
$$\partial_{\bar{t}} \tilde{\mathbf{v}} = -2\lambda \nabla \overline{V}(\tilde{\mathbf{a}}; \omega)$$

with

$$\tilde{\mathbf{a}}(0) = 0, \quad \tilde{\mathbf{v}}(0) = \mathbf{v}_0.$$

Corresponding Hamiltonian

$$H_{cl}(\tilde{\mathbf{a}}, \tilde{\mathbf{v}}) = \frac{\|\tilde{\mathbf{v}}\|^2}{2} + 2\lambda \overline{V}(\tilde{\mathbf{a}}).$$

Further assumptions on the potential

$\overline{V}$ is strongly mixing in the uniform sense.

Let

$$\varphi(\rho) = \sup\{|\mathbb{P}(A) - \mathbb{P}(B|A)|, \quad r > 0, \\ A \in \mathcal{C}^i_r,\ B \in \mathcal{C}^e_{r+\rho}\}, \rho > 0.$$

We assume

$$\sup_{\rho \geq 0} \rho^p \varphi(\rho) < \infty.$$

Two-point spatial correlation function

$$R(\tilde{\mathbf{a}}) = 4\mathbb{E}(\overline{V}(\tilde{\mathbf{a}})\overline{V}(0)).$$

We assume that

$$R \in C^\infty(\mathbb{R}^N)$$

such that

> $\widehat{R}$ does not vanish identically on
> any $H_{\mathbf{p}} = \{\tilde{\mathbf{v}} \in \mathbb{R}^N,\ \tilde{\mathbf{v}} \cdot \mathbf{p} = 0\}, \mathbf{p} \in \mathbb{R}^N$.

Diffusive regime

Macroscopic time over which diffusion is observed is

$$1/\lambda^2 < \overline{C}\epsilon |\log h|/\lambda.$$

As $\lambda \to 0$ and $h \to 0$, we need that $|\log h|\lambda \to \infty$ as $h \to 0$.

Sufficient condition:

$$\lambda = \frac{1}{|\log h|^{1-\alpha}},$$

$\alpha \in (0, 1)$.

Momentum diffusion

$(\underline{\mathbf{v}}(\bar{t}))_{\bar{t}\geq 0}$ generated by $\mathcal{L}$

$$\mathcal{L}u(\underline{\mathbf{v}}) = \sum_{i,j=1}^{N} \partial_{\underline{v}_i}(D_{ij}(\underline{\mathbf{v}})\partial_{\underline{v}_j}u(\underline{\mathbf{v}})).$$

Diffusion matrix

$$D_{ij}(\mathbf{k}) := -\frac{1}{2\|\mathbf{k}\|}\int_{-\infty}^{\infty} \partial_{x_i}\partial_{x_j}R(s\frac{\mathbf{k}}{\|\mathbf{k}\|})ds, \quad \mathbf{k} \in \mathbb{R}^N.$$

MOMENTUM DIFFUSION IN $N \geq 2$

Theorem 2

Suppose $\|\mathbf{v}_0\| \neq 0$, and that there exists $\tilde{\alpha} > 0$ such that the coupling constant $\lambda \to 0$ as $h \to 0$ with $|\log h|\lambda^{3/2+\tilde{\alpha}} \to \infty$. Then

$$(\lambda^2 \overline{\mathbf{a}}(\bar{t}/\lambda^2), \overline{\mathbf{v}}(\bar{t}/\lambda^2))_{\bar{t}\in(0,T)} \to (\int_0^{\bar{t}} \underline{\mathbf{v}}(s)ds, \underline{\mathbf{v}}(\bar{t}))_{\bar{t}\in(0,T)}$$

in law (weakly) as $h, \lambda \to 0$.

Still, momentum diffusion implies spatial diffusion over longer time scales !

(At least for $N \geq 3$.)

SPATIAL DIFFUSION IN $N \geq 3$

Liouville equation

$$\partial_t \phi^\lambda = \partial_{\bar{t}} \bar{\mathbf{a}}|_{\bar{\mathbf{a}}=\mathbf{x},\bar{\mathbf{v}}=\mathbf{k}} \cdot \nabla_{\mathbf{x}} \phi^\lambda + \partial_{\bar{t}} \bar{\mathbf{v}}|_{\bar{\mathbf{a}}=\mathbf{x},\bar{\mathbf{v}}=\mathbf{k}} \cdot \nabla_{\mathbf{k}} \phi^\lambda$$

with

$$\phi^\lambda(\mathbf{x}, 0, \mathbf{k}) = \phi_0(\lambda^{2+\beta}\mathbf{x}, \mathbf{k}), \beta > 0.$$

Spatial diffusion

$$\partial_t u = \sum_{i,j} d_{ij}(\|\mathbf{k}\|)\partial_{x_i}\partial_{x_j} u,$$

with

$$u(\mathbf{x}, 0, \mathbf{k}) = \frac{1}{\Gamma_{N-1}} \int_{S^{N-1}} \phi_0(\mathbf{x}, \|\mathbf{k}\|\mathbf{l}) d\Sigma(\mathbf{l}).$$

Here,

$$d_{ij}(\|\mathbf{k}\|) = \frac{1}{\Gamma_{N-1}} \int_{S^{N-1}} \|\mathbf{k}\| l_i \chi_j(\|\mathbf{k}\|\mathbf{l}) d\boldsymbol{\Sigma}(\mathbf{l}),$$

where χ_j are (mean-zero) solutions of

$$\sum_{i,j=1}^{N} \partial_{k_i}(D_{ij}(\mathbf{k})\partial_{k_j}\chi_j) = -\|\mathbf{k}\|\widehat{k}_j.$$

Theorem 3

Suppose $\|\mathbf{v}_0\| \neq 0$, and that there exists $\tilde{\alpha} > 0$ such that $\lambda \to 0$ as $h \to 0$ with $|\log h|\lambda^{1+\tilde{\alpha}} \to \infty$. Then there exists $\tilde{\beta} \in (0, \tilde{\alpha}/2)$ such that, for all $0 < \beta < \tilde{\beta}$,

$$\lim_{\lambda,h\to 0} \sup_{(t,\mathbf{x},\mathbf{k})\in[0,T]\times K} |\mathbb{E}[\phi^{\lambda}(\mathbf{x}/\lambda^{2+\beta}, t/\lambda^{2+2\beta}, \mathbf{k})] - u(\mathbf{x}, t, \mathbf{k})| = 0.$$

Previous results hold under more general assumptions.

GENERAL ASSUMPTIONS

(A1) *Energy.* $\exists$ a C^3-functional $F : H_1 \to \mathbb{R}$, such that $F'(.) = f(.)$,

$$\sup_{\|u\|_{H_1} \le M} \|F''(u)\|_{\mathcal{B}(H_1,H_{-1})} < \infty,$$

$$\sup_{\|u\|_{H_1} \le M} \|F'''(u)\|_{H_1 \to \mathcal{B}(H_1,H_{-1})} < \infty.$$

(A2)*Symmetry.* $F(\mathcal{T}.) = F(.)$, where $\mathcal{T}$ is

(i) $\mathcal{T}_a^{tr} : u(x) \to u(x-a), a \in \mathbb{R}^N$,

(ii) $\mathcal{T}_R^{r} : u(x) \to u(R^{-1}x), R \in SO(N)$,

(iii) $\mathcal{T}_\gamma^{g} : u(x) \to e^{i\gamma}u(x), \gamma \in [0, 2\pi)$,

(iv) $\mathcal{T}_v^{b} : u(x) \to e^{\frac{i}{2}v\cdot x}u(x), v \in \mathbb{R}^N$.

(A3)*Solitary wave solutions.* $\exists I \subset \mathbb{R}$ such that $\forall \mu \in I$, the nonlinear eigenvalue problem

$$(-\Delta + \mu)\eta_\mu - f(\eta_\mu) = 0$$

has a positive, spherically symmetric solution $\eta_\mu \in L^2 \cap C^2$, such that

$$\| |x|^3 \eta_\mu \|_{L^2} + \| |x|^2 |\nabla \eta_\mu| \|_{L^2} + \| |x|^2 \partial_\mu \eta_\mu \|_{L^2} < \infty,$$

for all $\mu \in I$. ($\eta_\mu(x) \alpha e^{-\sqrt{\mu}|x|}$ for large x)

(A4)*Orbital Stability.*

$$\partial_\mu \int dx \eta_\mu^2 > 0$$

for $\mu \in I$.

(A5) *Hessian.*

$$\mathcal{L}_\mu := -\Delta + \mu - f'(\eta_\mu)$$

The null space is

$$\mathcal{N}(\mathcal{L}_\mu) = span\{(0, \eta_\mu), (\partial_{x_n}\eta_\mu, 0), n = 1, \cdots, N\}.$$

TOOL BOX

Hamiltonian structure

Real inner product (Riemannian metric) on H^1

$$\langle u, v\rangle := \mathrm{Re}\int d\mathbf{x}\ u\overline{v}.$$

Symplectic 2-form

$$\Xi(u, v) := \mathrm{Im}\int d\mathbf{x}\ u\overline{\mathbf{v}} = \langle u, iv\rangle.$$

Hamiltonian functional

$$H_\lambda(\psi) := \frac{1}{2}\int d\mathbf{x}\ |\nabla\psi|^2 + \frac{\lambda}{2}\int V_h|\psi|^2 - F(\psi).$$

The Hamiltonian is nonautonomous.

$$\partial_t H_\lambda(\psi) = \frac{\lambda}{2} \int d\mathbf{x}\ (\partial_t V_h)|\psi|^2, \quad \mathbb{P} - \text{a.s.}$$

$$\partial_\mu m(\mu) > 0$$

$\Rightarrow \eta_\mu$ is a local minimizer of $H_{\lambda=0}(\psi)$ restricted to the balls

$$\mathcal{B}_m := \{\psi \in H^1 : N(\psi) = m\}.$$

Critical points of the action functional

$$\mathcal{E}_\mu(\psi) := \frac{1}{2}\int d\mathbf{x}\ (|\nabla\psi|^2 + \mu|\psi|^2) - F(\psi).$$

Soliton manifold

$$\mathcal{M}_s := \{\eta_\sigma := T_{\mathbf{av}\gamma}\eta_\mu,$$
$$\sigma = (\mathbf{a}, \mathbf{v}, \gamma, \mu) \in \mathbb{R}^N \times \mathbb{R}^N \times [0, 2\pi) \times I\}.$$

Combined transformation $T_{\mathbf{av}\gamma}$

$$\psi_{\mathbf{av}\gamma} := T_{\mathbf{av}\gamma}\psi = e^{i(\frac{1}{2}\mathbf{v}\cdot(\mathbf{x}-\mathbf{a})+\gamma)}\psi(\mathbf{x}-\mathbf{a}).$$

The tangent space to $\mathcal{M}_s$ at $\eta_\mu \in \mathcal{M}_s$

$$\mathcal{T}_{\eta_\mu}\mathcal{M}_s = span\{E_t, E_g, E_b, E_s\},$$

where

$$\begin{aligned}
E_t &:= \nabla_{\mathbf{a}} T^{tr}_{\mathbf{a}} \eta_\mu|_{\mathbf{a}=0} = -\nabla\eta_\mu \\
E_g &:= \partial_\gamma T^{g}_{\gamma} \eta_\mu|_{\gamma=0} = i\eta_\mu \\
E_b &:= 2\nabla_{\mathbf{v}} T^{b}_{\mathbf{v}} \eta_\mu|_{\mathbf{v}=0} = i\mathbf{x}\eta_\mu \\
E_s &:= \partial_\mu \eta_\mu.
\end{aligned}$$

Note that

$$
\begin{aligned}
&e_j := -\partial_{x_j},\ j = 1, \cdots, N, \\
&e_{j+N} := i x_j,\ j = 1, \cdots, N, \\
&e_{2N+1} := i, \\
&e_{2N+2} := \partial_\mu,
\end{aligned}
$$

when acting on $\eta_\sigma \in \mathcal{M}_s$, generate the basis vectors $\{e_\alpha \eta_\sigma\}_{\alpha=1}^{2N+2}$ of $\mathcal{T}_{\eta_\sigma}\mathcal{M}_s$.

Symplectic structure of the soliton manifold

$$\Xi_\sigma|_{\mathcal{T}_{\eta_\sigma}\mathcal{M}_s} := \{\langle e_\alpha \eta_\sigma, i e_\beta \eta_\sigma\rangle\}_{1\le\alpha,\beta\le 2N+2}$$

$$= \begin{pmatrix} 0 & -m(\mu)\mathbf{1}_{N\times N} & 0 & -\frac{1}{2}\mathbf{v}m'(\mu) \\ m(\mu)\mathbf{1}_{N\times N} & 0 & 0 & \mathbf{a}m'(\mu) \\ 0 & 0 & 0 & m'(\mu) \\ \frac{1}{2}\mathbf{v}^T m'(\mu) & -\mathbf{a}^T m'(\mu) & -m'(\mu) & 0 \end{pmatrix}.$$

Group structure

$\{e_\alpha\}_{\alpha=1,\cdots,2N+1}$ are generators of the Lie algebra $\mathfrak{g}$ corresponding to the Heisenberg group H^{2N+1}.

$$(\mathbf{a},\mathbf{v},\gamma)\cdot(\mathbf{a}',\mathbf{v}',\gamma') = (\mathbf{a}'',\mathbf{v}'',\gamma''),$$

$$\mathbf{a}'' = \mathbf{a} + \mathbf{a}',$$

$$\mathbf{v}'' = \mathbf{v} + \mathbf{v}',$$

and

$$\gamma'' = \gamma' + \gamma + \frac{1}{2}\mathbf{v}\cdot\mathbf{a}'.$$

Zero modes

$$i\mathcal{L}_\mu : \mathcal{T}_{\eta_\mu}\mathcal{M}_s \to \mathcal{T}_{\eta_\mu}\mathcal{M}_s$$

$$(i\mathcal{L}_\mu)^2 X = 0, \quad \forall X \in \mathcal{T}_{\eta_\mu}\mathcal{M}_s.$$

Skew-Orthogonal decomposition

Let

$$U_\delta := \{\psi \in H^1, \quad \inf_{\sigma\in\Sigma_0} \|\psi - \eta_\sigma\|_{H^1} \le \delta\}.$$

For $\delta \ll 1$ and $\forall \psi \in U_\delta$,

$$\exists! \sigma(\psi) \in C^1(U_\delta, \Sigma)$$

$$\Xi(\psi - \eta_{\sigma(\psi)}, X) = 0,$$

for all $X \in \mathcal{T}_{\eta_{\sigma(\psi)}} \mathcal{M}_s$.

PROOF OF EFFECTIVE DYNAMICS

Step 1: Reparametrized equations of motion

Using skew-orthogonal (or Lyapunov-Schmidt decomposition) $\Rightarrow$

$$\psi(t) = \eta_{\sigma(t)} + w(t) \quad \mathbb{P} - a.s.$$

Dynamics on the soliton manifold = Hamiltonian flow generated by the NLSE restricted to the soliton manifold $\Rightarrow$ reparametrized equations of motion for $\sigma(t)$.

Proposition

The parameter $\sigma = (\mathbf{a}, \mathbf{v}, \gamma, \mu)$ satisfy

$$\begin{aligned}
\partial_t a_j &= v_j + O(C_{|c|,w,h}) \\
\partial_t v_j &= -2\partial_{x_j} \lambda V_h(\mathbf{a}, t; \omega) + O(C_{|c|,w,h}) \\
\partial_t \gamma &= \mu + \frac{1}{4}\|\mathbf{v}\|^2 - \lambda V_h(\mathbf{a}, t; \omega) + O(C_{|c|,w,h}) \\
\partial_t \mu &= O(C_{|c|,w,h}),
\end{aligned}$$

where

$$C_{|c|,w,h} := \sup_{\omega \in \overline{\Omega}} \{|c| \|w\|_{H^1} + \lambda h^2 + \|w\|_{H^1}^2\}$$

and

$$|c| := \sup\{|\partial_t a_j - v_j|, |\partial_t v_j + 2\partial_{x_j} V_h|, \\ |\partial_t \gamma - \mu - \frac{1}{4}\|\mathbf{v}\| + \lambda V_h|, |\partial_t \mu|\}.$$

Step 2:Control of the fluctuation

Use an approximate Lyapunov functional and the coercivity property of the Hessian to control the H^1-norm of the fluctuation w.

Lyapunov functional

$$\mathcal{C}_\mu(u,v) := \mathcal{E}_\mu(u) - \mathcal{E}_\mu(v), \quad u,v \in H^1(\mathbb{R}^N).$$

Upper bound

$$\sup_{\omega\in\overline{\Omega}} |\partial_t \mathcal{C}_\mu(u,\eta_\mu)|$$
$$\leq C \sup_{\omega\in\overline{\Omega}} \left(\lambda h^2 \|w\|_{H^1} + (|c| + \lambda h + \|w\|_{H^1}^2)\|w\|_{H^1}^2\right).$$

Lower bound

$$\sup_{\omega\in\overline{\Omega}} |\mathcal{C}_\mu(u,\eta_\mu)| \geq \sup_{\omega\in\overline{\Omega}} \frac{\rho}{2}\|w\|_{H^1} - \overline{C} \sup_{\omega\in\overline{\Omega}} \|w\|^3_{H^1}.$$

Control of the fluctuation together with the reparametrized equations of motion give Theorem 1.

PROOF OF THEOREM 2

Explicit control of

effective dynamics of center of mass

"minus"

dynamics of classical particle

(using Gronwall Lemma).

Comparison with auxiliary dynamics

$$\partial_t \tilde{\mathbf{a}} = \tilde{\mathbf{v}}$$
$$\partial_t \tilde{\mathbf{v}} = -2\lambda \nabla \overline{V}(\tilde{\mathbf{a}};\omega)$$

with

$$\tilde{\mathbf{a}}(0) = 0, \quad \tilde{\mathbf{v}}(0) = \mathbf{v}_0.$$

Lemma

$$(\lambda^2 \overline{\mathbf{a}}(\bar{t}/\lambda^2), \overline{\mathbf{v}}(\bar{t}/\lambda^2))_{\bar{t}\in[0,T]}$$

converges $\mathbb{P}$-a.s. (strongly) to the stochastic process

$$(\lambda^2 \tilde{\mathbf{a}}(\bar{t}/\lambda^2), \tilde{\mathbf{v}}(\bar{t}/\lambda^2))_{\bar{t}\in[0,T]},$$

as $\lambda, h \to 0$.

Together with Theorem 1 and the results on momentum diffusion for classical particles, we have Theorem 2.

PROOF OF THEOREM 3

Explicit control of

effective dynamics of center of mass

"minus"

dynamics of classical particle

(using method of characteristics to solve the associated H-J equations).

Auxiliary spatial diffusion

$$\begin{aligned}\partial_t \tilde{\phi}^\lambda =& \partial_{\tilde{t}} \tilde{\mathbf{a}}|_{\tilde{\mathbf{a}}=\mathbf{x},\tilde{\mathbf{v}}=\mathbf{k}} \cdot \nabla_{\mathbf{x}} \tilde{\phi}^\lambda + \partial_{\tilde{t}} \tilde{\mathbf{v}}|_{\tilde{\mathbf{a}}=\mathbf{x},\tilde{\mathbf{v}}=\mathbf{k}} \cdot \nabla_{\mathbf{k}} \tilde{\phi}^\lambda \\ =& \, \mathbf{k} \cdot \nabla_{\mathbf{x}} \tilde{\phi}^\lambda - 2\lambda \nabla_{\mathbf{x}} \overline{V} \cdot \nabla_{\mathbf{k}} \tilde{\phi}^\lambda.\end{aligned}$$

Lemma

ϕ^λ and $\tilde{\phi}^\lambda$ $\in C^1(\mathbb{R}; C^1(\mathbb{R}^{2N}) \cap W^{1,\infty}(\mathbb{R}^{2N}))$ $\mathbb{P}-$a.s..

Lemma

$$\lim_{\lambda,h\to 0} \sup_{(t,\mathbf{x},\mathbf{k})\in[0,T]\times K} [\tilde{\phi}^{\lambda}(\mathbf{x}/\lambda^{2+\beta}, t/\lambda^{2+2\beta}, \mathbf{k}) - \phi^{\lambda}(\mathbf{x}/\lambda^{2+\beta}, t/\lambda^{2+2\beta}, \mathbf{k})] = 0$$

$\mathbb{P}-$ a.s..

Together with Theorem 1 and results on spatial diffusion for classical particles, we have Theorem 3.

I thank J. Fröhlich, I.M. Sigal and C. Sulem for enjoyable discussions on problems related to NLSE.

THANK YOU !