Diffusion for Coupled Map Lattices

Antti Kupiainen

joint work with J. Bricmont

Oslo 8.6.2009

Deterministic diffusion

Coupled dynamics Coupled chaos Local energy Diffusion CML

Coupling
Diffusion for ma

Random environment

Slow dynamics annealed Slow dynamics

quenched Linear problem

Result

Scaling Scaling limit

Scaling limit RG for maps

Fixed point
Trivial case

Assumption

......

How to derive diffusion from first principles?

Diffusion is related to global conservation laws

- Hamiltonian systems: Total energy is conseved
- ► Show: Local energy diffuses

Extended systems: # of degrees of freedom $\to \infty$:

- ▶ Subsystems indexed by $x \in \mathbb{Z}^d$
- ▶ Dynamics: $H = H_{subsystems} + H_{interaction}$
- $ightharpoonup H_{interaction} = 0$: energy E_x of subsystem at x conserved
- ▶ $H_{interaction} \neq 0$: show $E_x(t)$ diffuses

Coupled dynamics

Models: Coupled flows and Coupled maps

1. Coupled weakly nonlinear systems:

- \blacktriangleright $u(t,x), x \in \mathbb{Z}^d, \partial_t^2 u = (\Delta r)u \lambda u^3$
- ▶ Hard! Diffusion at time scale λ^{-2} might be provable

2. Conservative systems with noise

Lots of results

Replace noise by chaos:

3. Coupled chaotic systems

- Coupled billiards or Anosov systems
- Coupled maps with a local conservation law

Coupled chaotic systems

Coupled dynamics

Coupled chaos

Local energi Diffusion

Coupling
Diffusion for ma

Random

environment

Slow dynam

annealed

Slow dynamic

Linear proble

Result

Accumptions

Scaling

Scaling lin

RG for ma

Fixed poir

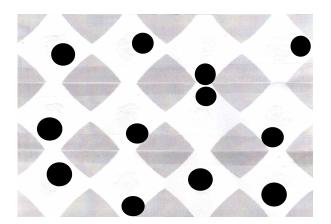
Trivial cas

Converger

Convergence

Hamiltonian au

Bunimovich, Liverani, Pellegrinotti, Suhov, Eckmann, Young ...



Coupled dynamics Coupled chaos

Local energy

CML

Diffusion for m

Random

environment

Slow dynami

annealed

quenched

Linear proble

Regult

nesuit

Scaling

Scaling lim

RG for ma

Fixed poi

Trivial cas

Assumptio

Hamiltonian system

Total energy is sum of **local energies** H_x , $x \in \mathbb{Z}^d$:

$$H = \sum_{x} H_{x}$$

No coupling: each H_x is conserved:

$$\dot{H}_x = 0$$

Turn on coupling: only H is conserved and

$$\dot{H}_{X} = -\nabla \cdot \mathbf{J}_{X}$$

 J_x flux of energy at site x.

Show: H_X diffuses and J_X is tied to to H_X by Fourier's law.

1. Initial condition

$$H_X(t=0) \to E \text{ as } |x| \to \infty.$$

Show:

$$H_X(t) = E + \mathcal{O}(t^{-d/2}e^{-\frac{|X|^2}{Dt}}).$$

2. Hydrodynamic limit: Initial condition

$$H_X(t=0) = \tau(\epsilon X)$$

Define

$$\tau(t,x) := \lim_{\epsilon \to 0} H_{x/\epsilon}(t/\epsilon^2) \ \ j(t,x) := \lim_{\epsilon \to 0} \frac{1}{\epsilon} J_{x/\epsilon}(t/\epsilon^2)$$

Show:

$$j = -\kappa(\tau)\nabla\tau$$
 Fourier law $\dot{\tau} = \nabla \cdot (\kappa(\tau)\nabla\tau)$ Diffusion

Coupled map lattice with a conservation law

Coupled dynam Coupled chaos Local energy Diffusion

CML Coupling

Diffusion for ma

Random

Slow dynam

Slow dynam annealed

Slow dynami quenched

Linear probl

Result

Assumption

Scaling

Scaling lin

Fixed poi

Trivial ca:

Assumptio

Convergence
Convergence
Hamiltonian syste

CML: Discrete space and time dynamics

- ▶ Subsystems indexed by $x \in \mathbb{Z}^d$
- ▶ Dynamical variables E_x , θ_x
- ▶ $E_x \in \mathbb{R}$ "Energy" of subsystem at site x

Subsystem dynamics

- ▶ Energy of each cell is **conserved**: $E_x \rightarrow E_x$
- ▶ θ_X chaotic variables $\theta_X \to f(\theta_X)$, f chaotic (hyperbolic)
- ▶ E.g. $\theta \in S^1$, $f(\theta) = 2\theta$

Perturb this dynamics so that **total energy** $\sum_{x} E_{x}$ is **conserved**.

Coupling

Coupled dynami Coupled chaos Local energy Diffusion CML

Coupling Diffusion for ma

Random

environment

Slow dynam

annealed

quenchea

Linear probl

Result

Cooling

Scaling lin

DC for me

Fixed po

Trivial ca

Assumpti

Converge

Hamiltonian sy

Coupling: nearby cells interact, exchange energy

$$E_X' = E_X + F_X(E, \theta)$$

 $\theta_X' = f(\theta_X) + g_X(E, \theta)$

where $E = (E_u)_{u \in \mathbb{Z}^d}$, $\theta = (\theta_u)_{u \in \mathbb{Z}^d}$. Demand:

- $ightharpoonup F_x, g_x$ depend on θ_u, E_u for u near x only
- $ightharpoonup \sum_{x} F_{x}(E, \theta) = 0$ for all E, θ . This is guaranteed by taking

$$F(E,\theta) = \nabla \cdot J(E,\theta)$$

 ∇ discrete gradient.

Then **total energy** $\sum_{x} E_{x}$ conserved.

Diffusion

Coupled dynamic Coupled chaos Local energy Diffusion CML Coupling

Diffusion for maps

Random environment

Slow dynamic

annealed Slow dynamic

Linear proble

Result

Scaling

Scaling limi

RG for ma

Fixed po

Trivial cas Assumption

Convergence Convergence

Hamiltonian system

Let, at t = 0, $E_x \to T$ as $|x| \to \infty$. Show $E_x(t)$ diffuses to T almost surely in $\theta(0)$

$$E_X(t) - T \sim t^{-d/2}e^{-x^2/\kappa t}$$

i.e.

$$L^d(E_{Lx}(L^2t)-T) \rightarrow e^{-x^2/\kappa t}$$
 as $L \rightarrow \infty$

Hydrodynamic scaling limit:

- ▶ Let $E_X(0) = \tau(\epsilon X)$
- ▶ Show: $\lim_{\epsilon \to 0} E(t/\epsilon^2, x/\epsilon) = \tau(t, x)$ satisfies

$$\dot{\tau} = \nabla \cdot (\kappa(\tau) \nabla \tau)$$

almost surely in $\theta(0)$.

Coupled dynamics Coupled chaos Local energy Diffusion

CML
Coupling
Diffusion for ma
Fast dynamics

Random

environmen

Slow dynam

Slow dynamics annealed

quenched

Linear probl

Result

Cooling

Scaling lin

DC for mo

Fixed poir

Trivial cas

Assumption

Convergen

Hamiltonian s

Dynamics of the chaotic variables

- ▶ Let $\theta_x \in S^1$, $x \in \mathbb{Z}^d$
- ▶ Let g_x depend only on θ :

$$\theta_{x}(t+1) = 2\theta_{x}(t) + g_{x}(\theta(t))$$

▶ *g* small, real analytic, local perturbation:

$$|\frac{\partial}{\partial \theta_y} g_x] \le \epsilon e^{-|x-y|}$$

Then θ -dynamics is **space time chaotic**.

Random environment

Coupled dynamics Coupled chaos Local energy Diffusion CML

Coupling
Diffusion for map

Random environment

Slow dynamic annealed Slow dynamic

Linear pro

Result

Assumpti

Scaling

Scaling li

RG for

Fixed p

Trivial c

Assumn

Assump

Converg

Hamiltonian s

Space time mixing dynamics

$$\mathbb{E}(F_{x}(\theta(t))G_{y}(\theta(0))) - \mathbb{E}(F_{x}(\theta(t))\mathbb{E}G_{y}(\theta(0)) \leq Ce^{-c(t+|x-y|)}$$

- ▶ \mathbb{E} be expectation in $m(d\theta(0))$
- ▶ *m* Lebesgue measure on $(S^1)^{\mathbb{Z}^d}$
- \triangleright F, G smooth, local functions of θ

Sampling $\theta(0)$ with m makes $\theta_x(t)$ random variables.

 $\theta_x(t)$ acts as **random environment** for the slow variables E.

Environment is **weakly correlated** in space and time.

Slow dynamics

Coupled dynamics Coupled chaos Local energy Diffusion

Coupling
Diffusion for map

Random

Slow dynamics

Slow dynam

annealed Slow dynam

quenched

Linear pi

Result

Accumptio

Scaling

Scaling lin

RG for ma

Fixed poi

Trivial cas

Assumptio

Converge

Hamiltonian sys

Dynamics of slow variables:

$$E_X(t+1) - E_X(t) = \nabla \cdot J_X(E(t), \theta(t))$$

with $\theta_x(t)$ random, weakly correlated in space and time Slow dynamics is a random nonlinear drift

To prove $E_X(t)$ diffuses a.s. in $\theta(0)$ amounts to prove quenched diffusion for $E_X(t)$.

Assume:

- ▶ $J_x(E,\theta)$ real analytic in E,θ
- Local, translation and rotation symmetric

Slow dynamics:

annealed

Consider first the **annealed** case, i.e. take average over θ :

$$E_X(t+1) - E_X(t) = \nabla \cdot \mathbb{E}[J_X(E(t),\cdot)] := \nabla \cdot \mathcal{J}_X(E(t)).$$

Then

$$\mathcal{J}_{x}(E) = 0$$
, E constant

Expand around *E* constant:

$$\mathcal{J}_{x}(E) = \sum_{y} \kappa(E)_{xy} \nabla E_{y}$$

by analyticity, isotropy and locality.

Annealed dynamics is a discrete nonlinear diffusion

$$E(t+1) - E(t) = \nabla \cdot \kappa(E(t)) \nabla E(t)$$

Slow dynamics: quenched

Coupled dynamic: Coupled chaos Local energy Diffusion CML

Coupling Diffusion for map

Random environment

Slow dynamic Slow dynamic

annealed Slow dynamics,

quenched

Linear proble

Assumptions

Scaling

Scaling lim

RG for ma

Fixed poi

Assumptio

Convergen

Hamiltonian syste

Let

$$\beta_{\mathsf{X}}(\mathsf{E}(t),t) = \mathsf{J}_{\mathsf{X}}(\mathsf{E}(t),\theta(t)) - \mathbb{E}[\mathsf{J}_{\mathsf{X}}(\mathsf{E}(t),\cdot)]$$

be the fluctuating part. Then slow dynamics becomes

$$E(t+1) - E(t) = \nabla \cdot \kappa(E(t)) \nabla E(t) + \nabla \cdot \beta(E(t), t)$$

i.e. nonlinear diffusion with random drift:

$$\mathbb{E}\beta(E,t)=0$$

Physically expect $\kappa(E(t))$ **positive** and hope for β to be a small perturbation.

Example: Linear problem

Linear problem

Suppose the slow dynamics is **linear** in *E*. Then

$$E_X(t+1) = \sum_y \rho_{XY}(t) E_Y(t)$$

with

$$\sum_{x} p_{xy}(t) = 1.$$

Suppose also $p_{xy} \geq 0$. Then

 $p_{xy}(t)$ are transition probabilities of a random walk

 $E_x(t)$ is (proportional to) the probability of finding the walker at x at time t

 $p_{xy}(t)$ space and time dependent, random i.e. **Random** walk in random environment

Prove quenched CLT for such walks

Random environment Slow dynamic

Slow dynamics annealed Slow dynamics

Linear pro

Result

Scaling Scaling lim

RG for map

Trivial case

Convergen

Convergence

Control nonlinear perturbation of RWRE

$$E(t+1) - E(t) = \nabla \cdot \kappa(E(t)) \nabla E(t) + \nabla \cdot \beta(E(t), t)$$

$$\kappa_{xy}(E) = \kappa(x - y) + k_{xy}(E)$$

with κ strictly positive operator, k, β small, local and analytic in $\|\Im E\|_{\infty} < \delta$.

Result. Let $E_x(0) \to 0$ as $t \to \infty$. Almost surely in $\theta(0)$

$$L^d E_{Lx}(L^2 t) \rightarrow C e^{-x^2/\kappa t}$$
 as $L \rightarrow \infty$

(in norm
$$||(1+|x|^{d+1})E||_{\infty}$$
)

Assumptions

Coupled dynamic
Coupled chaos
Local energy
Diffusion
CML
Coupling
Diffusion for maps
Fast dynamics

environment Slow dynamic

Slow dynamics: annealed

Linear pro

Result

Assumptions

Scaling lin

RG for ma

Fixed poir

Trivial cas

Assumptio

Convergence

Hamiltonian system

$$k_{xy}(E) = \sum_{A \subset \mathbb{Z}^d} k_{xyA}(E)$$

 $\beta_X(t, E) = \sum_{A,B \subset \mathbb{Z}^d} \beta_{xAB}(t, E)$

with k_{xyA} , β_{xAB} satisfying

- ▶ Analytic in $\|\Im E\|_{\infty} < \delta$
- $\blacktriangleright |\kappa_{XVA}| < \epsilon e^{-d(x,A)}$
- ▶ β_{xAB} , $\beta_{x'A'B'}$ independent if $B \cap B' = \emptyset$
- $ightharpoonup \mathbb{E}(eta_{xAB})^2 < \epsilon e^{-d(x,y,A)}$

Scaling

Coupled dynamic Coupled chaos Local energy Diffusion CML

Coupling
Diffusion for map

Random

Slow dynamics annealed

Slow dynamics quenched

Linear prob

Result Assumption

Assumption Scaling

Scaling lin

RG for ma

Fixed poin

Trivial case

Converge

Convergence Hamiltonian syste **Dynamics:** E(t+1,x) = f(t,x,E(t))

Scaling map S_L : $(S_L E)(x) = L^d E(Lx)$

Rescaled energies. Let L > 1

$$E_n(t) = S_{L^n} E(L^{2n} t).$$

These flow with renormalized dynamics

$$E_n(t+1)=f_n(t,E_n(t)).$$

with

$$f_n(t) = S_{L^n}(f(L^{2n}t + L^{2n} - 1) \circ \cdots \circ f(L^{2n}t))S_{L^{-n}}$$

Slow dynamics Slow dynamics annealed

Slow dynamics quenched

Linear proble

Result

Assumption Scaling

Scaling limit

Fixed point

Assumptio

Convergenc

Hamiltonian syst

Scaling limit for *E*:

$$\lim_{n\to\infty}L^{nd}E(L^{2n},L^nx)=\lim_{n\to\infty}E_n(x)$$

where

$$E_n(x) = E_n(1,x).$$

 E_n flow with the **Renormalization group flow**

$$E_{n+1} = S_L[f_n(L^2-1) \circ \cdots \circ f_n(1)(E_n)]$$

where f_n is the scale L^n renormalized dynamics.

Coupled dynamics Coupled chaos Local energy Diffusion

Coupling
Diffusion for may

Random environment

Slow dynamics Slow dynamics annealed

Slow dynamics quenched

Linear probl Result

Assumption Scaling

Scaling lim

RG for maps

Trivial case

Convergence
Convergence

The dynamics changes with scale as

$$f_{n+1} = \mathcal{R}f_n$$

with

$$\mathcal{R}f(t) = \mathcal{S}_L f(L^2(t+1)-1) \circ \cdots \circ f(L^2t) \mathcal{S}_{L-1}$$

 ${\cal R}$ is the **Renormalization group flow** in a space of random dynamical systems.

Fixed point

We prove: almost surely in $\theta(0)$ the renormalized maps converge

$$f_n = \mathcal{R}^n f \to f^*$$

where the fixed point is **nonrandom and linear**:

$$f^*(E) = e^{\kappa \Delta} E.$$

Moreover, the renormalized energies converge almost surely to the fixed point

$$E_n \rightarrow E^* = Ae^{-x^2/4\kappa}$$

which is the diffusive scaling limit. In other words:

- Noise is irrelevant
- ► Nonlinearity is irrelevant

Coupled dynamics Coupled chaos

Coupled chaos Local energy Diffusion CML

Diffusion for map

Random environment

Slow dynamics annealed Slow dynamics

quenched Linear problem

Result Assumption

Scaling Scaling limit

RG for map

Trivial ca

Assumptions
Convergence
Convergence

rioninioanty io inforcia

Coupling
Diffusion for maps

Random environment

Slow dynamics annealed

Slow dynamic quenched

Result

Assumption Scaling

Scaling lim RG for map

Trivial case

Assumption Converger

Hamiltonian syste

Consider linear problem

$$f(E,x) = \sum_{y} T(x-y)E(y)$$

Then $f_n = T_n E$ with

$$T_n(\cdot) = L^{nd} T^{*L^{2n}}(L^n \cdot)$$

i.e.

$$\hat{T}_n(k) = \hat{T}(L^{-n}k)^{L^{2n}} \sim e^{L^n a \cdot k - ck^2}$$

if
$$\hat{T}(k) = 1 + a \cdot k - ck^2 + o(k^2)$$
.

Drift term a is a **relevant** variable, in our case random and nonlinear.

Assumptions

Local energy
Diffusion
CML

Coupling
Diffusion for ma

Random environment

Slow dynamic

annealed Slow dynamics quenched

Linear proble

Result

Scaling

Scaling lim

RG for ma

Fixed poi

Assumptions

Convergence Convergence

Hamiltonian system

Recall our assumptions:

$$f(t, E) = (1 + \kappa \Delta)E + \sum_{A \subset \mathbb{Z}^d} \nabla \cdot \kappa_A(E) \nabla E + \sum_{A, B \subset \mathbb{Z}^d} \nabla \cdot \beta_{AB}(t, E)$$

with κ_{xyA} , β_{xAB} satisfying

- Analytic in $\|\Im E\|_{\infty} < \delta$
- $|\kappa_{xyA}| < \epsilon e^{-d(x,A)}$
- ▶ β_{xAB} , $\beta_{x'A'B'}$ independent if $B \cap B' = \emptyset$
- $ightharpoonup \mathbb{E}(\beta_{xAB})^2 < \epsilon e^{-d(x,y,A)}$

Convergence

Coupled dynamic Coupled chaos Local energy Diffusion CML

Coupling
Diffusion for maps

Random environment

Slow dynamics: Slow dynamics: annealed

Slow dynamics quenched

Linear prob

Assumption

Scaling Iim

RG for ma

Fixed poir

Trivial case Assumption

Convergence Convergence

Hamiltonian system

Then

$$f_n(t,E) = (1 + \kappa_n \Delta)E + \sum_{A \subset \mathbb{Z}^d} \nabla \cdot \kappa_A^n(E) \nabla E + \sum_{A,B \subset \mathbb{Z}^d} \nabla \cdot \beta_{AB}^n(t,E)$$

with κ_{xyA}^n , β_{xAB}^n satisfying

- ▶ Analytic in $\|\Im E\|_{\infty} < L^{nd}\delta$
- $\blacktriangleright |\kappa_{xvA}^n| < \epsilon_n e^{-d(x,A)}$
- ▶ β_{xAB}^n , $\beta_{x'A'B'}^n$ independent if $B \cap B' = \emptyset$
- $ightharpoonup \mathbb{E}(\beta_{xAB}^n)^2 < \epsilon_n e^{-d(x,y,A)}$

with $\epsilon_n \to 0$ as $n \to \infty$.

Convergence

Diffusion for maps
Fast dynamics
Random
environment
Slow dynamics
Slow dynamics:
annealed
Slow dynamics;
quenched
Linear problem
Result
Assumptions

Scaling limi

RG for map Fixed point

Assumption

Convergence

Hamiltonian systen

- ► Analyticity strip expands by L^d each step ⇒ perturbative region expands upon iteration.
 Nonlinearities in E irrelevant.
- ▶ Noise variance contracts i.e. noise irrelevant
- Need also large deviation estimate: Scale Lⁿ drift (noise) can be arbitarily large, but with small probability, going down with n

Hamiltonian systems

What kind of CML should model Hamiltonian systems?

- Rare configurations of E can slow down mixing of energies and θ dynamics
- Annealed system is probably not uniformly elliptic and random drift can create traps with long lifetimes

These issues can be studied with the RG.

Coupled dynamics Coupled chaos Local energy Diffusion CML

Coupling
Diffusion for map

Random environment Slow dynamic

Slow dynamics: annealed Slow dynamics, quenched Linear problem

Result Assumption

Scaling Scaling limit

RG for map

Trivial case

Assumptions Convergence

Hamiltonian systems