Diffusion for Coupled Map Lattices

Antti Kupiainen

joint work with J. Bricmont

Oslo
8.6.2009

Deterministic diffusion

How to derive **diffusion from first principles?**

Diffusion is related to global conservation laws

- **Hamiltonian** systems: **Total energy** is conseved
- Show: **Local** energy diffuses

Extended systems: $\#$ of degrees of freedom $\to \infty$:

- Subsystems indexed by $x \in \mathbb{Z}^d$
- Dynamics: $H = H_{subsystems} + H_{interaction}$
- $H_{interaction} = 0$: energy E_x of subsystem at x conserved
- $H_{interaction} \neq 0$: show $E_x(t)$ diffuses

Coupled dynamics

Models: **Coupled flows** and **Coupled maps**

1. Coupled weakly nonlinear systems:

- $u(t,x)$, $x \in \mathbb{Z}^d$, $\partial_t^2 u = (\Delta - r)u - \lambda u^3$
- Hard! Diffusion at time scale λ^{-2} might be provable

2. Conservative systems with noise

- Lots of results

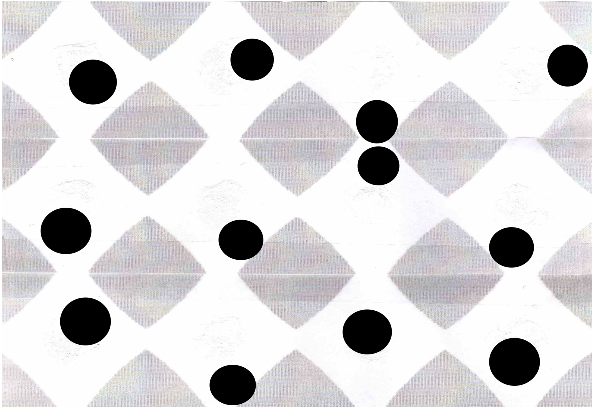
Replace noise by **chaos**:

3. Coupled chaotic systems

- Coupled billiards or Anosov systems
- Coupled maps with a local conservation law

Coupled chaotic systems

Bunimovich, Liverani, Pellegrinotti, Suhov, Eckmann, Young ...



Local energy and flux

Total energy is sum of **local energies** H_x, $x \in \mathbb{Z}^d$:

$$H = \sum_x H_x$$

No coupling: each H_x is conserved:

$$\dot{H}_x = 0$$

Turn on coupling: only H is conserved and

$$\dot{H}_x = -\nabla \cdot J_x$$

J_x **flux of energy** at site x.

Show: H_x **diffuses** and J_x is tied to to H_x by **Fourier's law**.

Diffusion

1. Initial condition

$$H_x(t=0) \to E \text{ as } |x| \to \infty.$$

Show:

$$H_x(t) = E + \mathcal{O}(t^{-d/2} e^{-\frac{|x|^2}{Dt}}).$$

2. Hydrodynamic limit: Initial condition

$$H_x(t=0) = \tau(\epsilon x)$$

Define

$$\tau(t,x) := \lim_{\epsilon \to 0} H_{x/\epsilon}(t/\epsilon^2) \quad j(t,x) := \lim_{\epsilon \to 0} \frac{1}{\epsilon} J_{x/\epsilon}(t/\epsilon^2)$$

Show:

$$j = -\kappa(\tau)\nabla\tau \quad \text{Fourier law}$$

$$\dot{\tau} = \nabla \cdot (\kappa(\tau)\nabla\tau) \quad \text{Diffusion}$$

Coupled map lattice with a conservation law

CML: **Discrete space and time** dynamics

- Subsystems indexed by $x \in \mathbb{Z}^d$
- Dynamical variables E_x, θ_x
- $E_x \in \mathbb{R}$ "Energy" of subsystem at site x

Subsystem dynamics

- Energy of each cell is **conserved**: $E_x \to E_x$
- θ_x chaotic variables $\theta_x \to f(\theta_x)$, f chaotic (hyperbolic)
- E.g. $\theta \in S^1$, $f(\theta) = 2\theta$

Perturb this dynamics so that **total energy** $\sum_x E_x$ is **conserved**.

Coupling

Coupling: nearby cells interact, exchange energy

$$\begin{aligned} E'_x &= E_x + F_x(E,\theta) \\ \theta'_x &= f(\theta_x) + g_x(E,\theta) \end{aligned}$$

where $E = (E_u)_{u\in\mathbb{Z}^d}$, $\theta = (\theta_u)_{u\in\mathbb{Z}^d}$. Demand:

- F_x, g_x depend on θ_u, E_u for u near x only
- $\sum_x F_x(E,\theta) = 0$ for all E, θ. This is guaranteed by taking
$$F(E,\theta) = \nabla \cdot J(E,\theta)$$
∇ discrete gradient.

Then **total energy** $\sum_x E_x$ conserved.

Diffusion

Let, at $t = 0$, $E_x \to T$ as $|x| \to \infty$.
Show $E_x(t)$ diffuses to T **almost surely** in $\theta(0)$

$$E_x(t) - T \sim t^{-d/2} e^{-x^2/\kappa t}$$

i.e.

$$L^d(E_{Lx}(L^2 t) - T) \to e^{-x^2/\kappa t} \text{ as } L \to \infty$$

Hydrodynamic scaling limit:

- Let $E_x(0) = \tau(\epsilon x)$
- Show: $\lim_{\epsilon \to 0} E(t/\epsilon^2, x/\epsilon) = \tau(t, x)$ satisfies

$$\dot{\tau} = \nabla \cdot (\kappa(\tau)\nabla\tau)$$

almost surely in $\theta(0)$.

Fast dynamics

Dynamics of the chaotic variables

- Let $\theta_x \in S^1$, $x \in \mathbb{Z}^d$
- Let g_x depend **only** on θ:

$$\theta_x(t+1) = 2\theta_x(t) + g_x(\theta(t))$$

- g small, real analytic, local perturbation:

$$|\frac{\partial}{\partial \theta_y} g_x] \leq \epsilon e^{-|x-y|}$$

Then θ-dynamics is **space time chaotic**.

Random environment

Space time mixing dynamics

$$\mathbb{E}(F_x(\theta(t))G_y(\theta(0))) - \mathbb{E}(F_x(\theta(t))\mathbb{E}G_y(\theta(0)) \leq Ce^{-c(t+|x-y|)}$$

- $\mathbb{E}$ be expectation in $m(d\theta(0))$
- m Lebesgue measure on $(S^1)^{\mathbb{Z}^d}$
- F, G smooth, local functions of θ

Sampling $\theta(0)$ with m makes $\theta_x(t)$ **random variables**.

$\theta_x(t)$ acts as **random environment** for the slow variables E.

Environment is **weakly correlated** in space and time.

Slow dynamics

Dynamics of slow variables:

$$E_x(t+1) - E_x(t) = \nabla \cdot J_x(E(t), \theta(t))$$

with $\theta_x(t)$ random, weakly correlated in space and time

Slow dynamics is a **random nonlinear drift**

To prove $E_x(t)$ **diffuses a.s. in** $\theta(0)$ amounts to prove **quenched** diffusion for $E_x(t)$.

Assume:

- $J_x(E, \theta)$ real analytic in E, θ
- Local, translation and rotation symmetric

Slow dynamics: annealed

Consider first the **annealed** case, i.e. take average over θ:

$$E_x(t+1) - E_x(t) = \nabla \cdot \mathbb{E}[J_x(E(t), \cdot)] := \nabla \cdot \mathcal{J}_x(E(t)).$$

Then

$$\mathcal{J}_x(E) = 0, \quad E \text{ constant}$$

Expand around E constant:

$$\mathcal{J}_x(E) = \sum_y \kappa(E)_{xy} \nabla E_y$$

by analyticity, isotropy and locality.

Annealed dynamics is a **discrete nonlinear diffusion**

$$E(t+1) - E(t) = \nabla \cdot \kappa(E(t)) \nabla E(t)$$

Slow dynamics: quenched

Let

$$\beta_x(E(t), t) = J_x(E(t), \theta(t)) - \mathbb{E}[J_x(E(t), \cdot)]$$

be the fluctuating part. Then slow dynamics becomes

$$E(t+1) - E(t) = \nabla \cdot \kappa(E(t)) \nabla E(t) + \nabla \cdot \beta(E(t), t)$$

i.e. **nonlinear diffusion** with **random drift:**

$$\mathbb{E}\beta(E, t) = 0$$

Physically expect $\kappa(E(t))$ **positive** and hope for β to be a small perturbation.

Example: Linear problem

Suppose the slow dynamics is **linear** in E. Then

$$E_x(t+1) = \sum_y p_{xy}(t) E_y(t)$$

with

$$\sum_x p_{xy}(t) = 1.$$

Suppose also $p_{xy} \geq 0$. Then

$p_{xy}(t)$ are **transition probabilities of a random walk**

$E_x(t)$ is (proportional to) the probability of finding the walker at x at time t

$p_{xy}(t)$ space and time dependent, random i.e. **Random walk in random environment**

Prove **quenched** CLT for such walks

Result

Control **nonlinear perturbation of RWRE**

$$E(t+1) - E(t) = \nabla \cdot \kappa(E(t))\nabla E(t) + \nabla \cdot \beta(E(t), t)$$

$$\kappa_{xy}(E) = \kappa(x-y) + k_{xy}(E)$$

with κ strictly positive operator, k, β small, local and analytic in $\|\Im E\|_\infty < \delta$.

Result. Let $E_x(0) \to 0$ as $t \to \infty$. Almost surely in $\theta(0)$

$$L^d E_{Lx}(L^2 t) \to C e^{-x^2/\kappa t} \quad \text{as} \quad L \to \infty$$

(in norm $\|(1+|x|^{d+1})E\|_\infty$)

Assumptions

$$k_{xy}(E) = \sum_{A \subset \mathbb{Z}^d} k_{xyA}(E)$$

$$\beta_x(t, E) = \sum_{A,B \subset \mathbb{Z}^d} \beta_{xAB}(t, E)$$

with k_{xyA}, β_{xAB} satisfying

- Analytic in $\|\Im E\|_\infty < \delta$
- $|\kappa_{xyA}| < \epsilon e^{-d(x,A)}$
- β_{xAB}, $\beta_{x'A'B'}$ **independent** if $B \cap B' = \emptyset$
- $\mathbb{E}(\beta_{xAB})^2 < \epsilon e^{-d(x,y,A)}$

Scaling

Dynamics: $E(t+1,x) = f(t,x,E(t))$

Scaling map S_L: $(S_L E)(x) = L^d E(Lx)$

Rescaled energies. Let $L > 1$

$$E_n(t) = S_{L^n} E(L^{2n} t).$$

These flow with **renormalized dynamics**

$$E_n(t+1) = f_n(t, E_n(t)).$$

with

$$f_n(t) = S_{L^n}(f(L^{2n}t + L^{2n} - 1) \circ \cdots \circ f(L^{2n}t))S_{L^{-n}}$$

Scaling limit

Scaling limit for E:

$$\lim_{n\to\infty} L^{nd} E(L^{2n}, L^n x) = \lim_{n\to\infty} E_n(x)$$

where

$$E_n(x) = E_n(1, x).$$

E_n flow with the **Renormalization group flow**

$$E_{n+1} = S_L[f_n(L^2 - 1) \circ \cdots \circ f_n(1)(E_n)]$$

where f_n is the scale L^n renormalized dynamics.

Renormalization group for for maps

The dynamics changes with scale as

$$f_{n+1} = \mathcal{R} f_n$$

with

$$\mathcal{R} f(t) = S_L f(L^2(t+1)-1) \circ \cdots \circ f(L^2 t) S_{L^{-1}}$$

$\mathcal{R}$ is the **Renormalization group flow** in a space of random dynamical systems.

Fixed point

We prove: **almost surely** in $\theta(0)$ the renormalized maps converge

$$f_n = \mathcal{R}^n f \to f^*$$

where the fixed point is **nonrandom and linear**:

$$f^*(E) = e^{\kappa\Delta} E.$$

Moreover, the renormalized energies converge almost surely to the fixed point

$$E_n \to E^* = Ae^{-x^2/4\kappa}$$

which is the diffusive scaling limit. In other words:

- Noise is irrelevant
- Nonlinearity is irrelevant

Trivial case

Consider linear problem

$$f(E,x) = \sum_y T(x-y)E(y)$$

Then $f_n = T_n E$ with

$$T_n(\cdot) = L^{nd} T^{*L^{2n}}(L^n \cdot)$$

i.e.

$$\hat{T}_n(k) = \hat{T}(L^{-n}k)^{L^{2n}} \sim e^{L^n a \cdot k - ck^2}$$

if $\hat{T}(k) = 1 + a \cdot k - ck^2 + o(k^2)$.

Drift term a is a **relevant** variable, in our case random and nonlinear.

Assumptions

Recall our assumptions:

$$f(t, E) = (1 + \kappa\Delta)E + \sum_{A \subset \mathbb{Z}^d} \nabla \cdot \kappa_A(E) \nabla E + \sum_{A,B \subset \mathbb{Z}^d} \nabla \cdot \beta_{AB}(t, E)$$

with κ_{xyA}, β_{xAB} satisfying

- Analytic in $\|\Im E\|_\infty < \delta$
- $|\kappa_{xyA}| < \epsilon e^{-d(x,A)}$
- β_{xAB}, $\beta_{x'A'B'}$ **independent** if $B \cap B' = \emptyset$
- $\mathbb{E}(\beta_{xAB})^2 < \epsilon e^{-d(x,y,A)}$

Convergence

Then

$$f_n(t,E) = (1+\kappa_n\Delta)E + \sum_{A\subset\mathbb{Z}^d} \nabla\cdot\kappa^n_A(E)\nabla E + \sum_{A,B\subset\mathbb{Z}^d} \nabla\cdot\beta^n_{AB}(t,E)$$

with κ^n_{xyA}, β^n_{xAB} satisfying

- Analytic in $\|\Im E\|_\infty < L^{nd}\delta$
- $|\kappa^n_{xyA}| < \epsilon_n e^{-d(x,A)}$
- β^n_{xAB}, $\beta^n_{x'A'B'}$ **independent** if $B\cap B' = \emptyset$
- $\mathbb{E}(\beta^n_{xAB})^2 < \epsilon_n e^{-d(x,y,A)}$

with $\epsilon_n \to 0$ as $n\to\infty$.

Convergence

- Analyticity strip expands by L^d each step $\Longrightarrow$ perturbative region expands upon iteration. **Nonlinearities in E irrelevant**.
- Noise variance contracts i.e. **noise irrelevant**
- Need also **large deviation estimate**: Scale L^n drift (noise) can be arbitarily large, but with small probability, going down with n

Hamiltonian systems

What kind of CML should model Hamiltonian systems?

- **Rare configurations** of E can **slow down** mixing of energies and θ dynamics
- Annealed system is probably not uniformly elliptic and random drift can create traps with long lifetimes

These issues can be studied with the RG.