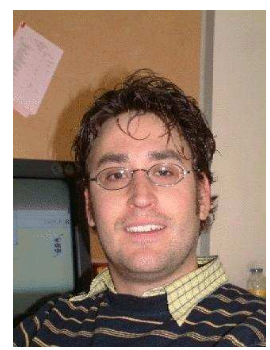
Leaky Repeated Interaction Quantum Systems *

Alain JOYE

* Joint work with



Laurent BRUNEAU (Université de Cergy) & Marco MERKLI (Memorial University)

The Formal Model

Quantum system $\mathcal{S}$:

- Finite dimensional system, driven by Hamiltonian $H_{\mathcal{S}}$ on $\mathfrak{H}_{\mathcal{S}}$, s.t. $\sigma(H_{\mathcal{S}}) = \{e_1, \cdots, e_d\}$.

Chain $\mathcal{C}$ of identical quantum sub-systems $\mathcal{E}_k \equiv \mathcal{E}$, $k = 1, 2, \cdots$:

$$\mathcal{C} = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4 + \cdots$$

- Each $\mathcal{E}_k$ is driven by the Hamiltonian $H_{\mathcal{E}\,k} \equiv H_{\mathcal{E}}$ on $\mathfrak{H}_{\mathcal{E}_k} \equiv \mathfrak{H}_{\mathcal{E}}$, $\dim \mathfrak{H}_{\mathcal{E}} \leq \infty$
- The chain $\mathcal{C}$ is driven by $H_{\mathcal{C}} \equiv H_{\mathcal{E}\,1} + H_{\mathcal{E}\,2} + \cdots$ on $\mathfrak{H}_{\mathcal{C}} \equiv \mathfrak{H}_{\mathcal{E}_1} \otimes \mathfrak{H}_{\mathcal{E}_2} \otimes \cdots$, with $[H_{\mathcal{E}\,j}, H_{\mathcal{E}\,k}] = 0, \;\; \forall j, k.$

Fermionic reservoir $\mathcal{R}$:

- ∞ -ly extended gas of indep. fermions at temperature β , driven by "$H_{\mathcal{R}}$" on "$\mathfrak{H}_{\mathcal{R}}$" .

The Formal Model

Complete system $\mathcal{S} + \mathcal{R} + \mathcal{C}$

- Formal Hilbert space $\mathfrak{H}_{\mathcal{S}} \otimes "\mathfrak{H}_{\mathcal{R}}" \otimes \mathfrak{H}_{\mathcal{E}}$

Interaction $\mathcal{S} - \mathcal{C}$

- $W_{\mathcal{S}\mathcal{E}}$ operator on $\mathfrak{H}_{\mathcal{S}} \otimes \mathfrak{H}_{\mathcal{E}_k}$, $k = 1, 2, \cdots$.

Interaction $\mathcal{S} - \mathcal{R}$

- $W_{\mathcal{S}\mathcal{R}}$ operator on $\mathfrak{H}_{\mathcal{S}} \otimes "\mathfrak{H}_{\mathcal{R}}"$.

The Formal Model

Complete system $\mathcal{S} + \mathcal{R} + \mathcal{C}$

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Interaction $\mathcal{S} - \mathcal{C}$

- $W_{\mathcal{S}\mathcal{E}}$ operator on $\mathfrak{H}_{\mathcal{S}} \otimes \mathfrak{H}_{\mathcal{E}_k}$, $k = 1, 2, \cdots$.

Interaction $\mathcal{S} - \mathcal{R}$

- $W_{\mathcal{S}\mathcal{R}}$ operator on $\mathfrak{H}_{\mathcal{S}} \otimes ''\mathfrak{H}_{\mathcal{R}}''$.

Evolution Let $\tau > 0$ be a duration, $\lambda = (\lambda_{\mathcal{R}}, \lambda_{\mathcal{E}}) \in \mathbb{R}^2$ be couplings

For $t = (m-1)\tau + s, \ \ 0 \leq s < \tau$,

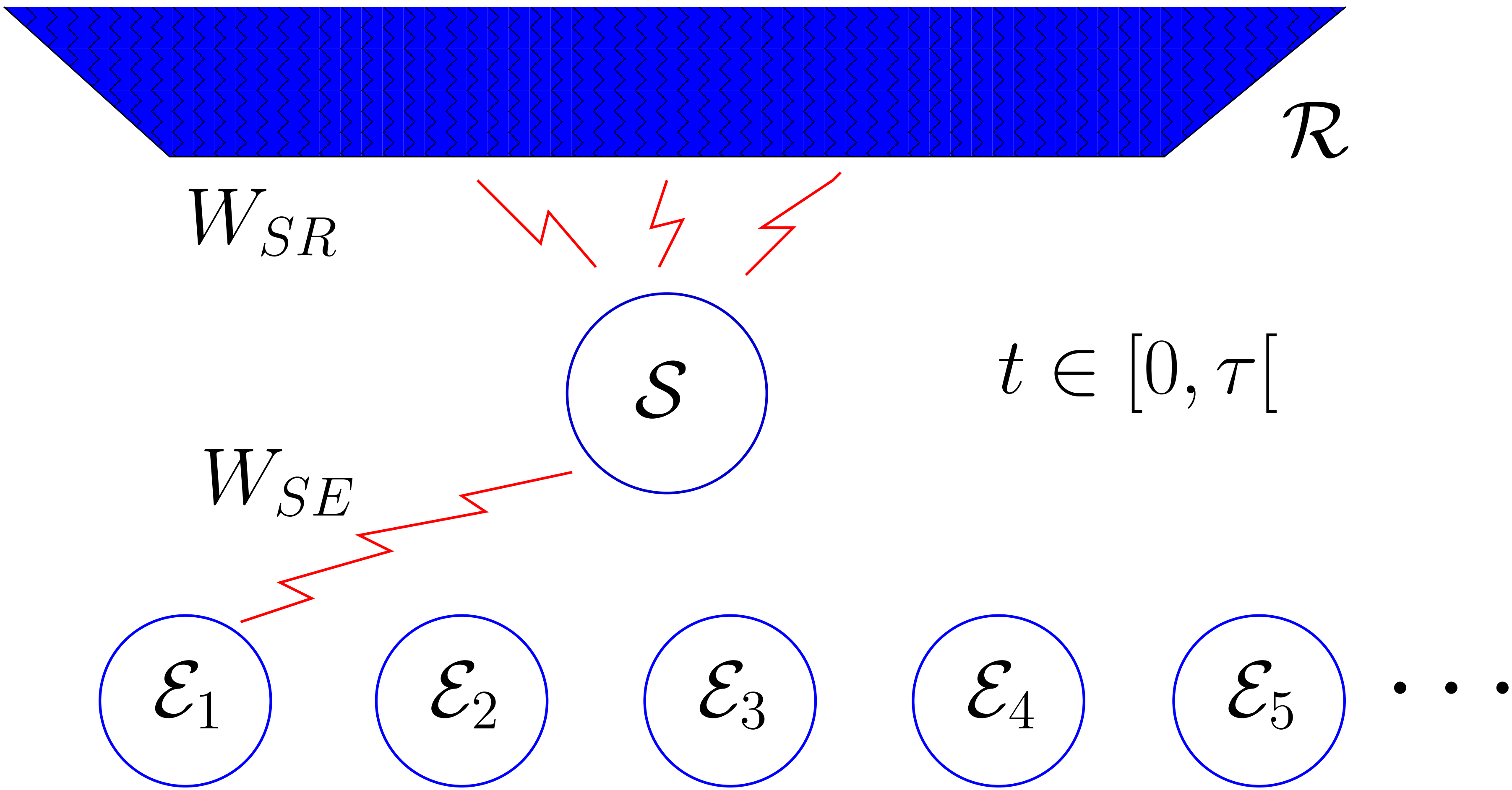
- $\mathcal{S}$, $\mathcal{R}$ and $\mathcal{E}_m$ are driven by $H_{\mathcal{S}} + ''H_{\mathcal{R}}'' + H_{\mathcal{E}} + \lambda_{\mathcal{R}} W_{\mathcal{S}\mathcal{R}} + \lambda_{\mathcal{E}} W_{\mathcal{S}\mathcal{E}}$
- $\mathcal{E}_k$ evolve freely with $H_{\mathcal{E}}$, $\forall\, k \neq m$

Leaky Repeated Interactions Quantum Systems

Pictorially

Leaky Repeated Interactions Quantum Systems

Pictorially



Leaky Repeated Interactions Quantum Systems

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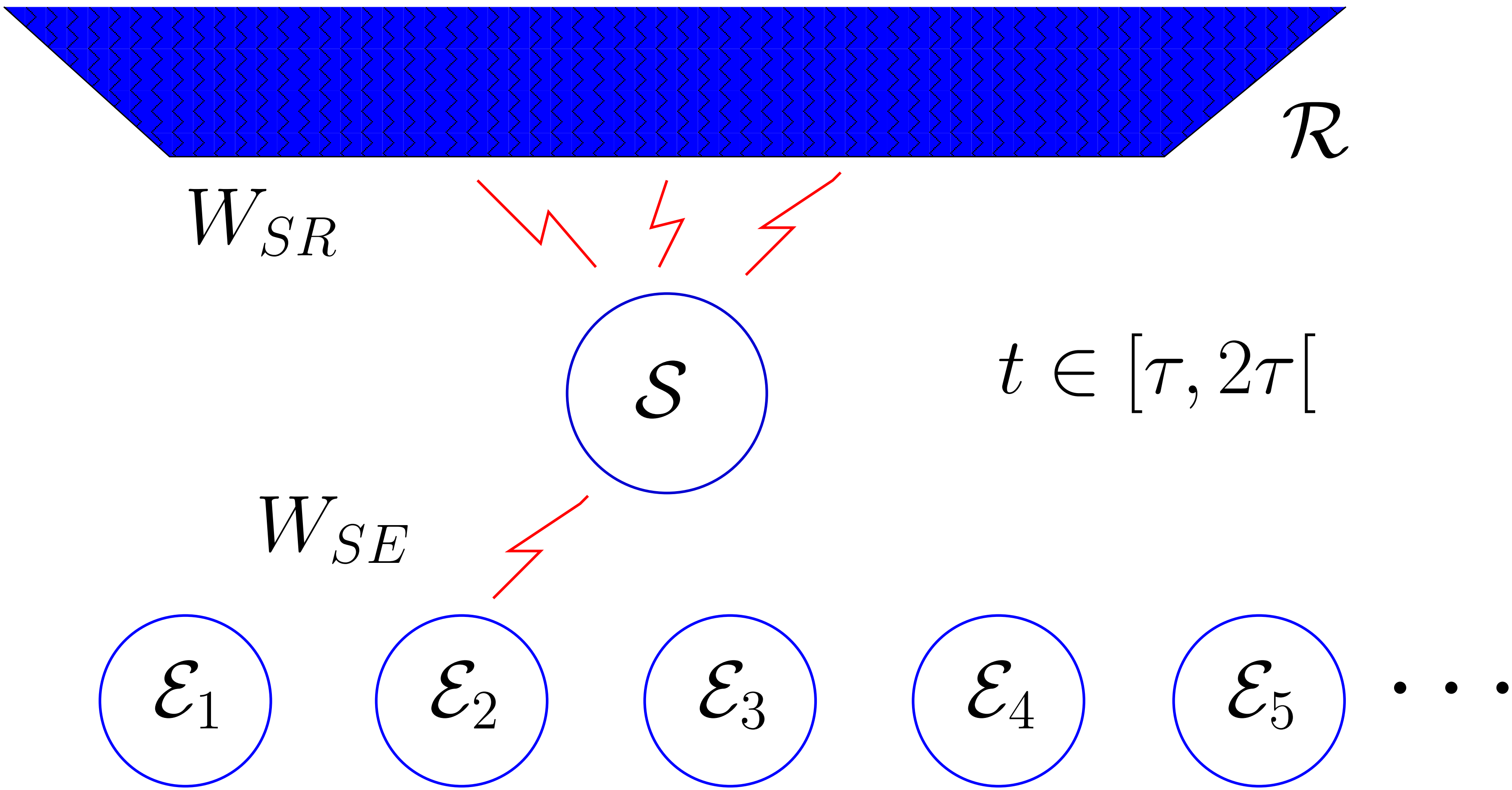

Leaky Repeated Interactions Quantum Systems

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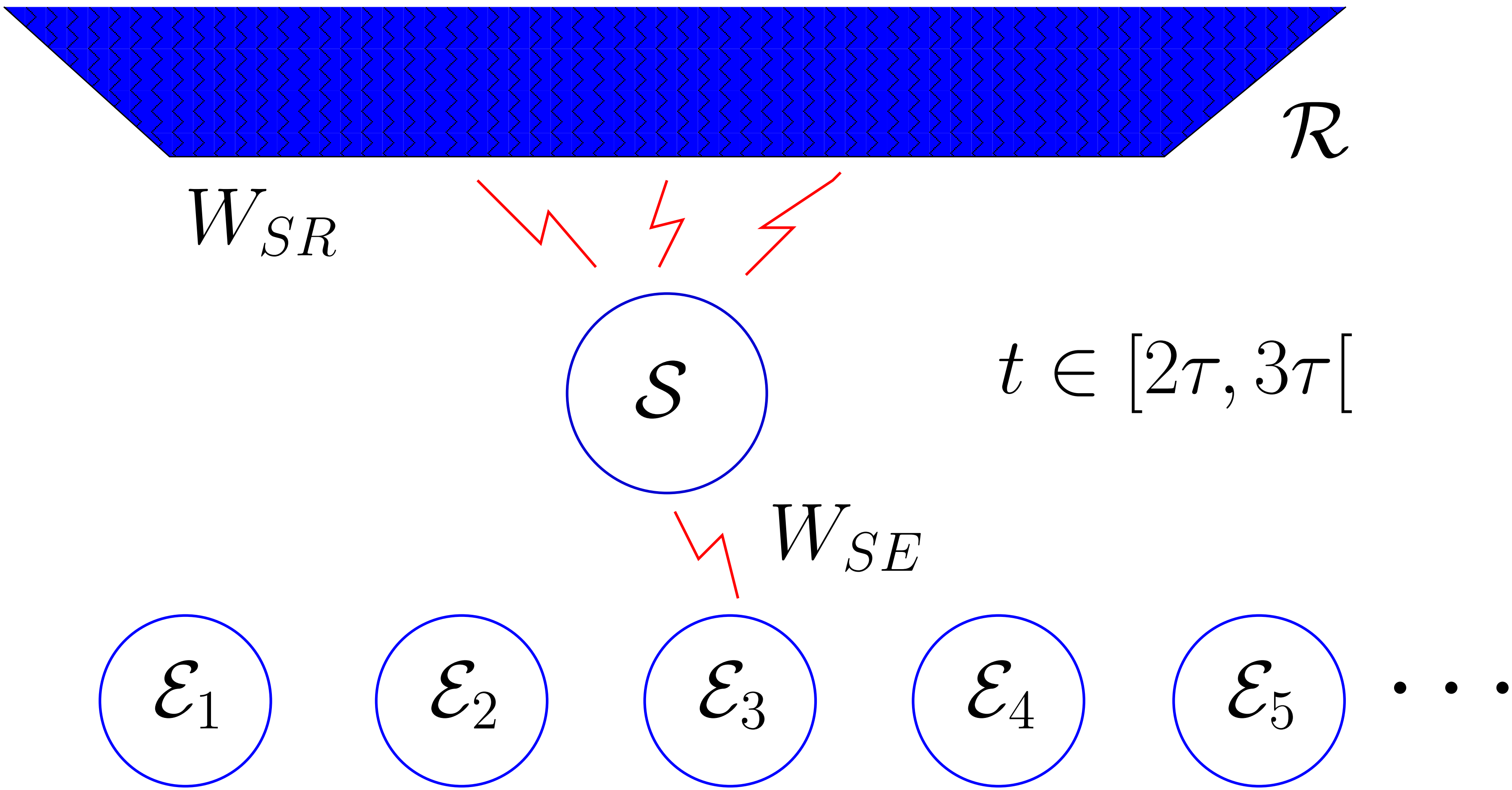
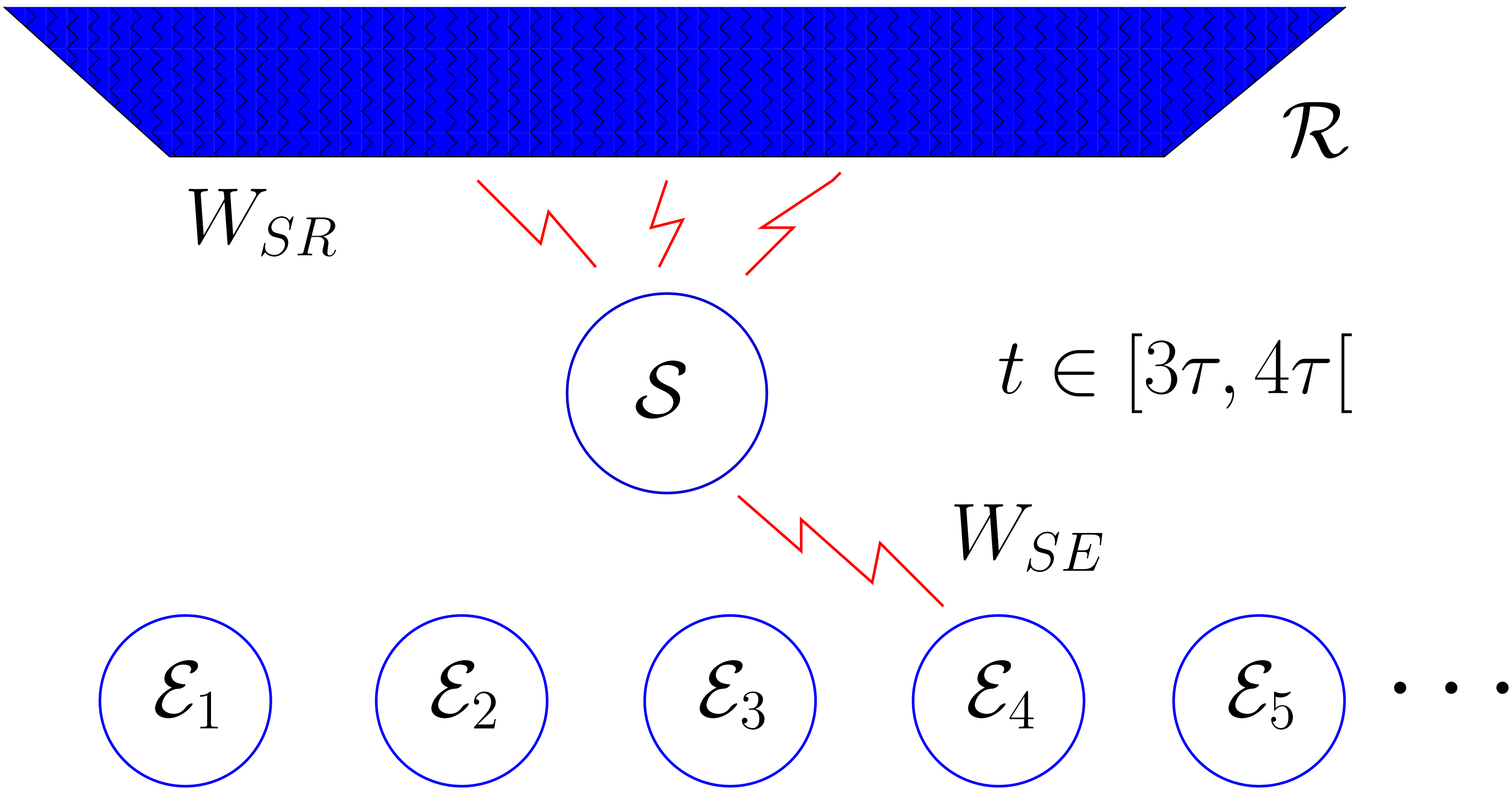


Leaky Repeated Interactions Quantum Systems

Pictorially



Questions

Large times asymptotics

Let $A = A_{SR} \otimes \mathbb{I}_{\mathcal{C}} \in \mathcal{B}(\mathfrak{H}_{\mathcal{S}} \otimes "\mathfrak{H}_{\mathcal{R}}" \otimes \mathfrak{H}_{\mathcal{C}})$ an observable acting on $\mathcal{S} + \mathcal{R}$

Let $\alpha^t(A)$ be its Heisenberg evolution, at time $t = m\tau$

Let $\rho : \mathcal{B}(\mathfrak{H}_{\mathcal{S}} \otimes "\mathfrak{H}_{\mathcal{R}}" \otimes \mathfrak{H}_{\mathcal{C}}) \to \mathbb{C}$ be a state (“density matrix”)

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Let $\rho : \mathcal{B}(\mathfrak{H}_{\mathcal{S}} \otimes "\mathfrak{H}_{\mathcal{R}}" \otimes \mathfrak{H}_{\mathcal{C}}) \to \mathbb{C}$ be a state ("density matrix")

- Existence of $\lim_{m\to\infty} \rho \circ \alpha^{m\tau}(A) = \rho^+(A)$?

  Dependence of $\rho^+(A)$ on the coupling constants $\lambda = (\lambda_{\mathcal{R}}, \lambda_{\mathcal{E}})$?
- Exchanges between $\mathcal{R}$ and $\mathcal{C}$ through $\mathcal{S}$?

  Energy variations, Entropy production, 2 nd law of thermodynamics ?
- Non-trivial examples ?

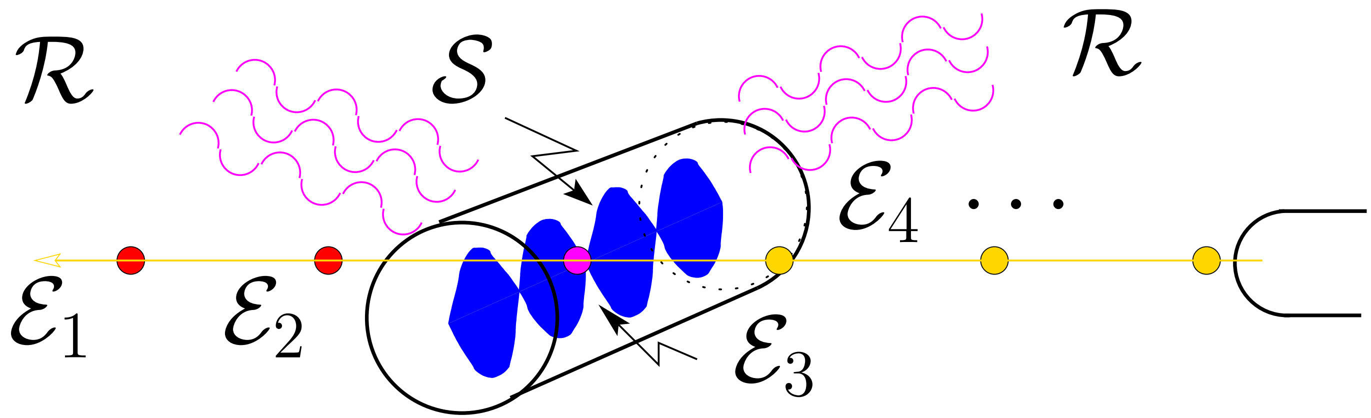
Remark :

If $\lambda_{\mathcal{R}} = 0$, then $\mathcal{S} + \mathcal{C} \Rightarrow$ convergence to a NESS Bruneau-J.-Merkli 06

If $\lambda_{\mathcal{E}} = 0$, then $\mathcal{S} + \mathcal{R} \Rightarrow$ return to equilibrium Jaksic-Pillet 96

Motivation

One-atom maser — Walther et al ’85, Haroche et al ’92



- $\mathcal{S}$: one mode of E-M field in a cavity
- $\mathcal{E}_k$: atom $\#k$ interacting with the mode
- $\mathcal{C}$: sequence of atoms passing through the cavity
- $\mathcal{R}$: environment responsible for losses

Ideal RIQS used as simple models — Vogel et al 93, Wellens et al 00, BJM 06, Bruneau Pillet 09

Random RIQS to model fluctuations — BJM 08

Leaky RIQS to account for losses

GNS Representation

Density matrix on $\mathfrak{H}$ $\quad\longrightarrow\quad$ pure state on $\mathcal{H} = \mathfrak{H} \otimes \mathfrak{H}$:

$$
\begin{array}{lrcl}
\text{state} & \rho = \sum \lambda_j |\varphi_j\rangle\langle\varphi_j| & \longrightarrow & \Psi_\rho = \sum \sqrt{\lambda_j}\varphi_j \otimes \varphi_j \\
\text{observable} & A \in \mathcal{B}(\mathfrak{H}) & \longrightarrow & \Pi(A) = A \otimes \mathbb{I}_{\mathfrak{H}} \in \mathcal{B}(\mathcal{H}) \\
\text{so that} & \mathrm{Tr}_{\mathfrak{H}}(\rho A) & = & \mathrm{Tr}_{\mathcal{H}}(|\Psi_\rho\rangle\langle\Psi_\rho|\Pi(A))
\end{array}
$$

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\end{array}
$$

Dynamics

$$
\begin{array}{rcl}
A \in \mathcal{B}(\mathfrak{H}) & \mapsto & \alpha^t(A) = e^{itH} A e^{-itH} \in \mathcal{B}(\mathfrak{H}) \\
\rho \in \mathcal{B}_1(\mathfrak{H}) & \mapsto & e^{-itH} \rho e^{itH} \in \mathcal{B}_1(\mathfrak{H})
\end{array}
$$

Liouville operator

Given ρ invariant, $\exists$ a unique self-adjoint L on $\mathcal{H} = \mathfrak{H} \otimes \mathfrak{H}$ s.t.

$$
\left\{
\begin{array}{rcl}
\Pi(\alpha^t(A)) & = & e^{itL}\Pi(A)e^{-itL} \in \mathcal{H} \\
L\Psi_\rho & = & 0
\end{array}
\right.
$$

Simple setup

$$
L = H \otimes \mathbb{I}_{\mathfrak{H}} - \mathbb{I}_{\mathfrak{H}} \otimes H
$$

GNS for Fermi Bath

Araki-Wyss 64

Temperature β^{-1}

Ingredients: $\tilde{\mathfrak{h}} = L^2(\mathbb{R}^+, \mathfrak{G})$ one particle Hilbert, $\mathfrak{G}$ auxil. Hilbert sp.

one particle Hamiltonian $\tilde{h}$ s.t.

$(\tilde{h}\tilde{f})(s) = s\tilde{f}(s)$, $s \in \mathbb{R}^+$, $\forall \tilde{f} \in \tilde{\mathfrak{h}} = L^2(\mathbb{R}^+, \mathfrak{G})$

Hamiltonian $d\Gamma_-(\tilde{h})$ on Fock sp. $\Gamma_-(\tilde{\mathfrak{h}}) = \oplus_{n=0}^{\infty} \Gamma_-^n(\tilde{\mathfrak{h}})$

$a(\tilde{g})$, $a^*(\tilde{g})$ annih. and creat. op's on $\Gamma_-(\tilde{\mathfrak{h}})$, $\tilde{g} \in \tilde{\mathfrak{h}}$

Equilibrium State ω_β characterized by

$\omega_\beta(a^*(\tilde{g})a(\tilde{f})) = \langle \tilde{f} | (1 + e^{\beta\tilde{h}})^{-1} \tilde{g} \rangle$ and

$$\omega_\beta(a^*(\tilde{g}_n) \cdots a^*(\tilde{g}_1) a(\tilde{f}_1) \cdots a(\tilde{f}_n)) = \det(\omega_\beta(a^*(\tilde{g}_i) a(\tilde{f}_j)))$$

GNS for Fermi Bath

Araki-Wyss 64 + Jaksic-Pillet Gluing 02:

Enlarged Hilbert space $\mathcal{H}_{\mathcal{R}} = \Gamma_{-}(\mathfrak{h})$, $\mathfrak{h} = L^2(\mathbb{R}, \mathfrak{G})$

Liouvillean $L_{\mathcal{R}} = d\Gamma(h)$, with h s.t.

$(hf)(s) = sf(s)$, $s \in \mathbb{R}$, $\forall f \in \mathfrak{h} = L^2(\mathbb{R}, \mathfrak{G})$

Creat., annih. op's $a^*(g_\beta)$, $a(g_\beta)$, where $g_\beta \leftrightarrow \tilde{g}$ via

$$g_\beta(s) = (e^{-\beta s} + 1)^{-1/2} g(s)\,,\; g(s) = \begin{cases} \tilde{g}(s) & \text{if } s \geq 0 \\ \bar{\tilde{g}}(-s) & \text{if } s < 0. \end{cases}$$

Equilibrium State $|\Psi_R\rangle\langle\Psi_{\mathcal{R}}|$, $\Psi_{\mathcal{R}}$ vacuum of $\Gamma_{-}(\mathfrak{h})$

Note

$$\text{“}L^2(\mathbb{R}^+, \mathfrak{G}) + L^2(\mathbb{R}^+, \mathfrak{G}) = L^2(\mathbb{R}, \mathfrak{G})\text{”}$$

Formalization

After GNS (writing A for $\Pi(A)$)

- Hilbert spaces $\mathcal{H}_\mathcal{S}$, $\mathcal{H}_\mathcal{R}$, $\mathcal{H}_{\mathcal{E}_k}$, and $\mathcal{H}_\mathcal{C} = \mathcal{H}_{\mathcal{E}_1} \otimes \mathcal{H}_{\mathcal{E}_2} \otimes \mathcal{H}_{\mathcal{E}_3} \otimes \cdots$
- Algebras of observables $\mathfrak{M}_\# \subset \mathcal{B}(\mathcal{H}_\#)$, $\# = \mathcal{S}, \mathcal{R}, \mathcal{E}, \mathcal{C}$
- States on $\mathcal{S}, \mathcal{R}, \mathcal{E}, \mathcal{C}$ are density matrices on $\mathcal{H}_\mathcal{S}$, $\mathcal{H}_\mathcal{R}$, $\mathcal{H}_\mathcal{E}$, $\mathcal{H}_\mathcal{C}$
- Evolution of observables $A_\mathcal{S} \mapsto \alpha^t_\mathcal{S}(A_\mathcal{S})$, $A_\mathcal{R} \mapsto \alpha^t_\mathcal{R}(A_\mathcal{R})$, $A_\mathcal{E} \mapsto \alpha^t_\mathcal{E}(A_\mathcal{E})$

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- Assumption: $\exists$ invariant states (cyclic and separating)

$\Psi_\mathcal{S} \in \mathcal{H}_\mathcal{S}$, $\Psi_\mathcal{R} \in \mathcal{H}_\mathcal{R}$ and $\Psi_\mathcal{E} \in \mathcal{H}_\mathcal{E}$ s.t.

$$\alpha^t_\#(A_\#) = e^{itL_\#} A_\# e^{-itL_\#}, \text{ and } L_\# \Psi_\# = 0, \quad \text{where } \# = \mathcal{S}, \mathcal{R} \text{ and } \mathcal{E}$$

Formalization

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$\Psi_\mathcal{S} \in \mathcal{H}_\mathcal{S}$, $\Psi_\mathcal{R} \in \mathcal{H}_\mathcal{R}$ and $\Psi_\mathcal{E} \in \mathcal{H}_\mathcal{E}$ s.t.

$\alpha^t_\#(A_\#) = e^{itL_\#} A_\# e^{-itL_\#}$, and $L_\# \Psi_\# = 0$, where $\# = \mathcal{S}, \mathcal{R}$ and $\mathcal{E}$

System $\mathcal{S} + \mathcal{R} + \mathcal{C}$

On $\mathcal{H} = \mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{R} \otimes \mathcal{H}_\mathcal{C}$, driven by $L_{\text{free}} = L_\mathcal{S} + L_\mathcal{R} + \sum_k L_{\mathcal{E}_k}$

Interactions

$V_{S\#} \in \mathfrak{M}_\mathcal{S} \otimes \mathfrak{M}_\#$, the GNS repres. of $W_{S\#}$, $\# = \mathcal{R}, \mathcal{E}$ + tech. hyp.

Dynamics

Repeated interaction Schrödinger dynamics

For any $m \in \mathbb{N}$, if $t = m\tau$ and $\psi \in \mathcal{H}$,

$$U(m)\psi := e^{-i\tilde{L}_m} e^{-i\tilde{L}_{m-1}} \cdots e^{-i\tilde{L}_1} \psi$$

where the generator for the duration τ is

$$\tilde{L}_m = \tau L_m + \tau \sum_{k \neq m} L_{\mathcal{E},k}$$

with

$$\begin{cases} L_m = L_{\mathcal{S}} + L_{\mathcal{R}} + L_{\mathcal{E}} + V_m & \text{on } \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}} \otimes \mathcal{H}_{\mathcal{E}_m} \quad \text{coupled} \\ V_m = \lambda_{\mathcal{R}} V_{\mathcal{S}\mathcal{R}} + \lambda_{\mathcal{E}} V_{\mathcal{S}\mathcal{E}} & \\ L_{\mathcal{E},k} = L_{\mathcal{E}} & \text{on } \mathcal{H}_{\mathcal{E}_k} \quad \text{free} \end{cases}$$

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To be studied

Let $\varrho \in \mathcal{B}_1(\mathcal{H})$ be a state on $\mathcal{H}$ and $A_{\mathcal{S}\mathcal{R}} \in \mathfrak{M}$ an observable on $\mathcal{S} + \mathcal{R}$

$$m \mapsto \varrho(U^*(m) A_{\mathcal{S}\mathcal{R}} U(m)) \equiv \varrho(\alpha^{m\tau}(A_{\mathcal{S}\mathcal{R}})), \quad \text{as} \quad m \to \infty$$

Reduction to a Product of Operators

Special state

$\varrho_0 = \langle \Psi_0 | \cdot \Psi_0 \rangle$ where $\Psi_0 = \Psi_{\mathcal{SR}} \otimes \Psi_{\mathcal{C}}$ with

$\Psi_{\mathcal{SR}} = \Psi_{\mathcal{S}} \otimes \Psi_{\mathcal{R}} \in \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}} \equiv \mathcal{H}_{\mathcal{SR}}$ and

$\Psi_{\mathcal{C}} = \Psi_{\mathcal{E}_1} \otimes \Psi_{\mathcal{E}_2} \otimes \cdots \in \mathcal{H}_{\mathcal{C}}$

$P = \mathbb{I}_{\mathcal{H}_{\mathcal{SR}}} \otimes |\Psi_{\mathcal{C}}\rangle\langle\Psi_{\mathcal{C}}|$ is the projector on $\mathcal{H}_{\mathcal{SR}} \otimes \mathbb{C}\Psi_{\mathcal{C}} \simeq \mathcal{H}_{\mathcal{SR}}$

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$C-$ Liouvillean Given $L_{\mathcal{S}} + L_{\mathcal{R}}$, $L_{\mathcal{E}}$ and $V_m \in \mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{R}} \otimes \mathfrak{M}_{\mathcal{E}_m}$,

$$\exists \quad K_m \text{ s.t.} \begin{cases} e^{i\tilde{L}_m} A e^{-i\tilde{L}_m} = e^{iK_m} A e^{-iK_m} & \forall A \in \mathfrak{M}_{\mathcal{SR}} \otimes \mathfrak{M}_{\mathcal{C}} \\ K_m \Psi_{\mathcal{SR}} \otimes \Psi_{\mathcal{C}} = 0. \end{cases}$$

K_m is not self-adjoint, not even normal ! Jaksic, Pillet '02

Reduction to a Product of Operators

Special state

$$\varrho_0 = \langle\Psi_0| \cdot \Psi_0\rangle \quad \text{where} \quad \Psi_0 = \Psi_{\mathcal{SR}} \otimes \Psi_{\mathcal{C}} \quad \text{with}$$

$$\Psi_{\mathcal{SR}} = \Psi_{\mathcal{S}} \otimes \Psi_{\mathcal{R}} \in \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}} \equiv \mathcal{H}_{\mathcal{SR}} \quad \text{and}$$

$$\Psi_{\mathcal{C}} = \Psi_{\mathcal{E}_1} \otimes \Psi_{\mathcal{E}_2} \otimes \cdots \in \mathcal{H}_{\mathcal{C}}$$

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$C-$ Liouvillean   Given $L_{\mathcal{S}} + L_{\mathcal{R}}$, $L_{\mathcal{E}}$ and $V_m \in \mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{R}} \otimes \mathfrak{M}_{\mathcal{E}_m}$,

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K_m is not self-adjoint, not even normal !   Jaksic, Pillet '02

$$K_m = \tau(L_{\text{free}} + V_m - V_m'), \quad V_m' = J_m \Delta_m^{\frac{1}{2}} V_m \Delta_m^{-\frac{1}{2}} J_m$$

$$:= \tau(L_{\text{free}} + \tilde{V}_m)$$

Tomita-Takesaki '57

Reduction to a Product of Matrices

Evolution of ϱ_0

$$
\begin{aligned}
\varrho_0(\alpha^{m\tau}(A_{SR})) &= \langle \Psi_0 | e^{i\tilde{L}_1} \cdots e^{i\tilde{L}_m} A_{SR} e^{-i\tilde{L}_m} \cdots e^{-i\tilde{L}_1} \Psi_0 \rangle \\
&= \langle \Psi_0 | e^{iK_1} \cdots e^{iK_m} A_{SR} e^{-iK_m} \cdots e^{-iK_1} \Psi_0 \rangle \\
&= \langle \Psi_0 | P e^{iK_1} \cdots e^{iK_m} A_{SR} P \Psi_0 \rangle \\
&= \langle \Psi_0 | (P e^{iK_1} P)(P e^{iK_2} P) \cdots (P e^{iK_m} P) A_{SR} \Psi_0 \rangle \\
&\equiv \langle \Psi_{SR} | M_1 M_2 \cdots M_m A_{SR} \Psi_{SR} \rangle \\
&= \langle \Psi_{SR} | M^m A_{SR} \Psi_{SR} \rangle
\end{aligned}
$$

where $M_j \simeq P e^{iK_j} P$ on $\mathcal{H}_{SR}$ are all identical.

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&= \langle \Psi_0 | e^{iK_1} \cdots e^{iK_m} A_{SR} e^{-iK_m} \cdots e^{-iK_1} \Psi_0 \rangle \\
&= \langle \Psi_0 | P e^{iK_1} \cdots e^{iK_m} A_{SR} P \Psi_0 \rangle \\
&= \langle \Psi_0 | (P e^{iK_1} P)(P e^{iK_2} P) \cdots (P e^{iK_m} P) A_{SR} \Psi_0 \rangle \\
&\equiv \langle \Psi_{SR} | M_1 M_2 \cdots M_m A_{SR} \Psi_{SR} \rangle \\
&= \langle \Psi_{SR} | M^m A_{SR} \Psi_{SR} \rangle
\end{aligned}
$$

where $M_j \simeq P e^{iK_j} P$ on $\mathcal{H}_{SR}$ are all identical.

Reduced Dynamical Operators

$M \in \mathcal{B}(\mathcal{H}_{\mathcal{SR}})$ s.t.

$$
\begin{cases}
M \Psi_{\mathcal{SR}} = \Psi_{\mathcal{SR}} \\
\| M^n \varphi \| \leq C(\varphi), \quad \forall n \in \mathbb{N}, \quad \forall \varphi \text{ in a dense set}
\end{cases}
$$

Note: Ψ_{SR} cyclic and evolution is unitary.

Spectral Properties of RDO's

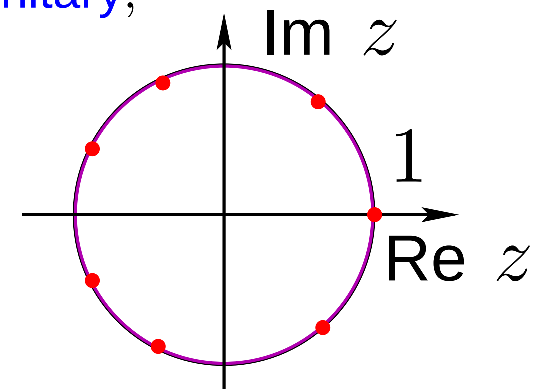
RDO

$$M = M_{(\lambda_{\mathcal{R}}, \lambda_{\mathcal{E}})}$$

Uncoupled case

$$M_{(0,0)} = e^{i\tau(L_{\mathcal{S}} + L_{\mathcal{R}})}$$ unitary,

- eigenvalues of $M_{(0,0)}$: $\{e^{i\tau(e_k - e_l)}\}_{k,l}$
- 1 is $\dim \mathfrak{h}_{\mathcal{S}}$-fold degenerate
- ess spec $M_{(0,0)} = \mathbb{S}^1$



Spectral Properties of RDO's

RDO

$$M = M_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}$$

Uncoupled case

$$M_{(0,0)} = e^{i\tau(L_{\mathcal{S}}+L_{\mathcal{R}})} \text{ unitary,}$$

- eigenvalues of $M_{(0,0)}$: $\{e^{i\tau(e_k - e_l)}\}_{k,l}$
- 1 is $\dim \mathfrak{h}_{\mathcal{S}}$-fold degenerate
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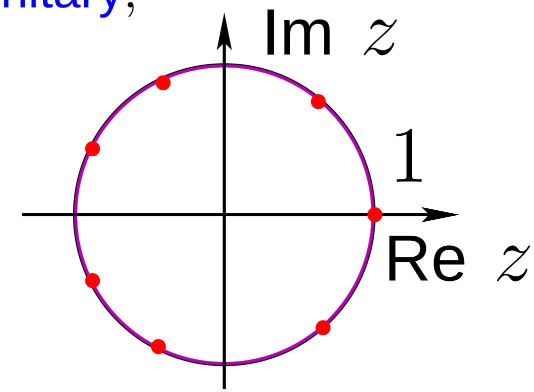

$(\lambda_{\mathcal{R}}, \lambda_{\mathcal{E}}) \neq (0,0)$ $\Rightarrow$ Perturbation of embedded eigenvalues

$L_{\mathcal{R}} = d\Gamma(h)$ with h mult. by s on $L^2(\mathbb{R}, \mathcal{G})$ is suitable for translation analyticity

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Translation Analyticity

Translation Group

$$\mathbb{R} \ni \theta \mapsto T(\theta) = \Gamma(e^{-\theta\partial_s}) \text{ on } \Gamma_-(L^2(\mathbb{R}, \mathcal{G}))$$

$$\text{s.t.} \quad (e^{-\theta\partial_s} f)(s) = f(s - \theta), \quad \forall f \in L^2(\mathbb{R}, \mathcal{G})$$

Assumption (A)

$\mathbb{R} \ni \theta \mapsto \tilde{V}_{\mathcal{SR}}(\theta) := T(\theta)^{-1}\tilde{V}_{\mathcal{SR}}T(\theta)$ admits an analytic extension to $\kappa_{\theta_0} = \{z \in \mathbb{C} \mid 0 < \text{Im } z < \theta_0\}$

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Recall

$$M = P \exp(iK) P\,, \quad \text{where}$$

$$K = \tau(L_0 + \lambda_{\mathcal{R}} \tilde{V}_{\mathcal{SR}} + \lambda_{\mathcal{E}} \tilde{V}_{\mathcal{SE}})\ , \ \ L_0 = L_{\mathcal{S}} + L_{\mathcal{R}} + L_{\mathcal{E}}$$

Theorem The following op's are analytic $\forall \theta \in \kappa_{\theta_0}$

$$K(\theta) = \tau(L_0 + \theta N + \lambda_{\mathcal{R}} \tilde{V}_{\mathcal{SR}}(\theta) + \lambda_{\mathcal{E}} \tilde{V}_{\mathcal{SE}}) \text{ on } D(L_0) \cap D(N),$$

$$M(\theta) = P \exp(iK(\theta)) P \in \mathcal{B}(\mathcal{H}_{\mathcal{SR}})$$

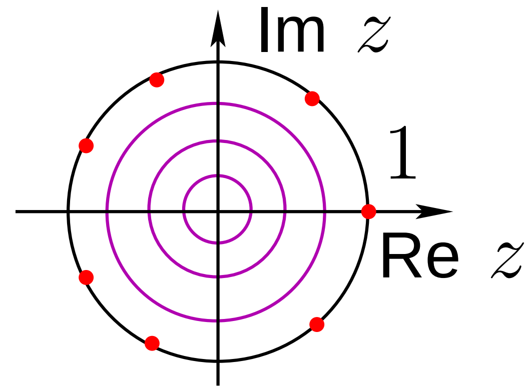
Translation Analyticity

Consequences

Discrete e.v. of $M_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(\theta)$ are θ-independent

Spectrum of $M_{(0,0)}(\theta) = \exp(i\tau(L_{\mathcal{S}} + L_{\mathcal{R}} + \theta N))$

- eigenvalues of $M_{(0,0)}(\theta)$: $\{e^{i\tau(e_k - e_l)}\}_{k,l}$
- 1 is $\dim \mathfrak{h}_{\mathcal{S}}$-fold degenerate
- ess spec $M_{(0,0)}(\theta) = \cup_{n=1}^{\infty}\{|z| = e^{-n\tau \text{Im}\theta}\}$



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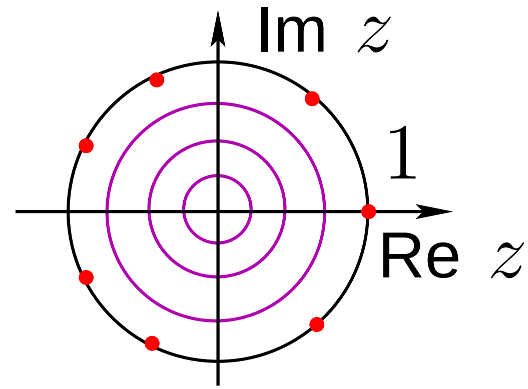
$$\left\{ \begin{array}{l} \text{eigenvalues of } M_{(0,0)}(\theta): \quad \{e^{i\tau(e_k - e_l)}\}_{k,l} \\ 1 \text{ is } \dim \mathfrak{h}_{\mathcal{S}}\text{-fold degenerate} \\ \text{ess spec } M_{(0,0)}(\theta) = \cup_{n=1}^{\infty}\{|z| = e^{-n\tau \mathrm{Im}\theta}\} \end{array} \right.$$

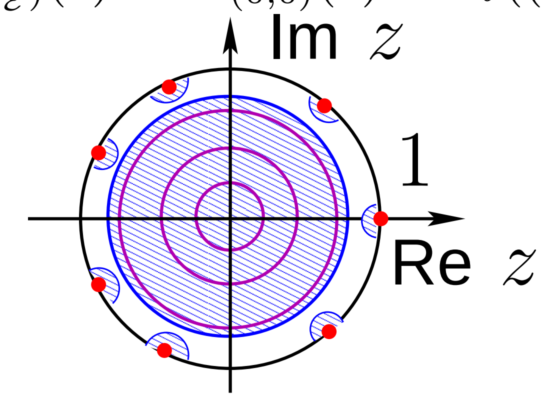


Perturbative approach

$$M_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(\theta) = M_{(0,0)}(\theta) + O_{\theta}((\lambda_{\mathcal{R}},\lambda_{\mathcal{E}}))$$

Lemma

$$\|(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})\| < \lambda_0(\theta) \ \Rightarrow \sigma(M_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(\theta)):$$



Asymptotic State

Analytic observables

$A_{\mathcal{SR}}$ s.t. $A_{\mathcal{SR}}(\theta) = T(\theta)^{-1} A_{\mathcal{SR}} T(\theta)$ analytic in κ_{θ_0}

Note: For $A_{\mathcal{SR}}$ analytic,

$$
\begin{aligned}
\varrho_0(\alpha^{m\tau}(A_{SR})) &= \langle \Psi_{SR} | M^m A_{SR} \Psi_{SR} \rangle \\
&= \langle \Psi_{SR} | M(\theta)^m A_{SR}(\theta) \Psi_{SR} \rangle
\end{aligned}
$$

Asymptotic State

Analytic observables

$$A_{\mathcal{SR}} \text{ s.t. } A_{\mathcal{SR}}(\theta) = T(\theta)^{-1} A_{\mathcal{SR}} T(\theta) \text{ analytic in } \kappa_{\theta_0}$$

Note: For $A_{\mathcal{SR}}$ analytic,

$$\begin{aligned} \varrho_0(\alpha^{m\tau}(A_{SR})) &= \langle \Psi_{SR} | M^m A_{SR} \Psi_{SR} \rangle \\ &= \langle \Psi_{SR} | M(\theta)^m A_{SR}(\theta) \Psi_{SR} \rangle \end{aligned}$$

Assumption (FGR)

$$\exists \theta_1 \in \kappa_{\theta_0}, \lambda_0(\theta_1) > 0 \text{ s.t. } \|(\lambda_{\mathcal{R}}, \lambda_{\mathcal{E}})\| < \lambda_0(\theta_1) \text{ implies}$$

$$\sigma(M_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(\theta_1)) \cap \mathbb{S} = \{1\} \text{ and } 1 \text{ is simple}$$

Consequences

$$\lim_{n\to\infty} M_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(\theta_1)^n = P_{1,M_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(\theta_1)} = |\Psi_{\mathcal{SR}}\rangle\langle\psi^*_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(\theta_1)|$$

Main Result

Theorem

Assume (A) and (FRG). For any state ϱ on $\mathcal{H}_{\mathcal{SR}} \otimes \mathcal{H}_{\mathcal{C}}$ and any analytic observable A_{SR}

$$\begin{aligned} \lim_{n\to\infty} \varrho(\alpha^{\tau n}_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(A_{SR})) &= \langle \psi^*_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(\theta_1) | A_{SR}(\theta_1) \Psi_{SR} \rangle \\ &\equiv \rho^+_{(\lambda_{\mathcal{R}},\lambda_{\mathcal{E}})}(A_{SR}). \end{aligned}$$

Instantaneous Observables

$\exists$ Upgrade to more general observables !

Application

- $\mathcal{S}$ and $\mathcal{E}$ spins with e.v. $\{0, E_\mathcal{S}\}$, resp. $\{0, E_\mathcal{E}\}$
- $\mathcal{R}$ Fermi gas at $\beta_\mathcal{R}$, equil. state $\omega_{\beta_\mathcal{R}}$
- $W_{\mathcal{S}\mathcal{E}} = a_S \otimes a_E^* + a_S^* \otimes a_E$
- $\Psi_\mathcal{S}$ tracial, $\Psi_\mathcal{E} \simeq \omega_{\beta,\mathcal{E}} = e^{-\beta_\mathcal{E} H_\mathcal{E}}/Z_{\beta_\mathcal{E}}$
- $W_{\mathcal{S}\mathcal{R}} = \sigma_x \otimes (a_R^*(f) + a_R(f))$, $f \in L^2(\mathbb{R}^+, \mathcal{G})$ "regular".

Perturbation theory

1) If $\|f(\sqrt{E_\mathcal{S}})\| > 0$ and $\tau(E_\mathcal{S} - E_\mathcal{E}) \neq 2\pi\mathbb{Z}^*$, then (FGR) holds

2) The asymptotic state ω_+ is given by

$$\omega_+ = (\gamma\, \omega_{\beta_\mathcal{R},\mathcal{S}} + (1-\gamma)\, \omega_{\tilde{\beta}_\mathcal{E},\mathcal{S}}) \otimes \omega_{\beta_\mathcal{R}} + \mathcal{O}_{\theta_1,\beta_\mathcal{R},\cdots}(\|(\lambda_\mathcal{R}, \lambda_\mathcal{E})\|)$$

with

$$\gamma = \frac{\lambda_\mathcal{R}^2 \pi \sqrt{E_\mathcal{S}}\|f(\sqrt{E_\mathcal{S}})\|^2}{\lambda_\mathcal{R}^2 \pi \sqrt{E_\mathcal{S}}\|f(\sqrt{E_\mathcal{S}})\|^2 + \lambda_\mathcal{E}^2 \tau \text{sinc}^2(\tau(E_\mathcal{S} - E_\mathcal{E})/2)}, \qquad \tilde{\beta}_\mathcal{E} = \beta_\mathcal{E} \frac{E_\mathcal{E}}{E_\mathcal{S}}.$$

Energy

$\alpha^m(\tilde{L}_m)$ is (the GNS rep. of) "total energy" for times $t \in [(m-1)\tau, m\tau)$.

Variation between $(m+1)\tau$ and $m\tau$,

$$\Delta E^{tot}(m) \quad = \quad \alpha^{m+1}(\tilde{L}_{m+1}) - \alpha^m(\tilde{L}_m) = \alpha^m(V_{m+1} - V_m)$$

Similarly

$$\begin{aligned} \Delta E^S(m) &= \alpha^{m+1}(L_{\mathcal{S}}) - \alpha^m(L_{\mathcal{S}}) \\ \Delta E^R(m) &= \alpha^{m+1}(L_{\mathcal{R}}) - \alpha^m(L_{\mathcal{R}}) \\ \Delta E^C(m) &= \alpha^{m+1}(L_{\mathcal{E}_{m+1}}) - \alpha^m(L_{\mathcal{E}_m}) \end{aligned}$$

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Asymptotic energy variation per unit time

$$dE_+^{\#} := \lim_{N\to\infty} \rho\left(\frac{\sum_{m=1}^N \Delta E^{\#}(m)}{N}\right)$$ exists under (A) and (FRG)

Property

$$dE_+^S = 0, \quad dE_+^{tot} = dE_+^R + dE_+^C$$

Entropy production

Let $\Psi_{\mathcal{S}}$ and $\Psi_{\mathcal{E}}$ correspond to Gibbs states at temperatures $\beta_{\mathcal{S}}$ and $\beta_{\mathcal{E}}$

Relative entropy ϱ and ϱ_0 are states on $\mathfrak{M}$, generalization of

$$Ent(\varrho|\varrho_0) = \text{Tr}\,(\rho(\ln\rho - \ln\rho_0)) \geq 0$$ Araki '75

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Variation of relative entropy w.r.t. KMS states Jaksic, Pillet '03

Let ϱ_0 correspond to $\Psi_{\mathcal{S}} \otimes \Psi_{\mathcal{R}} \otimes \Psi_{\mathcal{C}}$ and ϱ be any state,

$$\begin{aligned} \Delta S(m) &:= Ent(\varrho \circ \alpha^m|\varrho_0) - Ent(\varrho|\varrho_0) \\ &= \varrho(\alpha^m[\beta_{\mathcal{S}} L_{\mathcal{S}} + \beta_{\mathcal{R}} L_{\mathcal{R}} + \beta_{\mathcal{E}} \sum_{j=1}^{m} L_{\mathcal{E},j}] - \beta_{\mathcal{S}} L_{\mathcal{S}} - \beta_{\mathcal{R}} L_{\mathcal{R}} - \sum_{j=1}^{m} \beta_{\mathcal{E}_j} L_{\mathcal{E},j}) \end{aligned}$$

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Asymptotic entropy production rate

$$dS^+ := \lim_{N\to\infty} \frac{\Delta S(N)}{N}$$ exists and

$$dS^+ = \beta_{\mathcal{E}} dE_+^{\mathcal{C}} + \beta_{\mathcal{R}} dE_+^{\mathcal{R}}$$ 2^{nd} law