The SSEE

Discrete mode

Camalinatan

Invariances and large deviations of the current for the SSEP on $\ensuremath{\mathbb{Z}}$

arXiv:0902.2364

A. Gerschenfeld & B. Derrida

Laboratoire de Physique Statistique École Normale Supérieure (Paris)

THE SSI

Dynamics

Geometrie

Hydrodynamic the Known results

Discrete mod

The simple exclusion process

Microscopic dynamics

$$p q (= 1/2)$$

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Dynamics

Geometrie

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Discrete mod

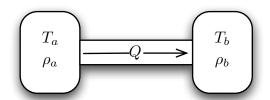
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The simple exclusion process

Microscopic dynamics

$$p q (= 1/2)$$

Simple transport model



 \Rightarrow Distribution of Q?

The SSE

Dynamics

Geometries

Hydrodynamic the

Discrete mod

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Geometries

Open system



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Dynamic

Geometries

Hydrodynamic theo

Discrete mod

Hydrodynamic

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Geometries

Open system



Fluctuations on a ring



Invariances and large deviations of the current for the SSEP on Z

A. Gerschenfeld

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Hydrodynamic the

Known results

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Hydrodynamic

Geometries

Open system



Fluctuations on a ring



Step initial condition on ${\mathbb Z}$



SSEP: Spohn (1989), Tracy Widom (2007-9)

TASEP: Schutz (1993), Johansson (2000), Prahofer Spohn (2000)

Dynamic

Hydrodynamic theory

Discrete mod

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Hydrodynamic theory

Variational principle

Kipnis Olla Varadhan 89, Spohn 91, Bertini DeSole Gabrielli Jona-Lasinio Landim 01

- Continuous model for diffusive systems
- Conserved particle density $\rho(x,t)$
- $\rho(x,t)$ diffuses on average, with excursions :

$$\log \mathbb{P}[\rho_0(x) \to \rho_T(x)] \sim \max_{\rho(x,t)} - \int_0^T dt \int dx \frac{(j + D(\rho)\partial_x \rho)^2}{2\sigma(\rho)}$$

- j(x, t) : current
- Conservation : $\partial_x i + \partial_t \rho = 0$
- On average, $j = -D(\rho)\partial_x \rho$ (Fick's law)

Discrete mod

Hydrodynamic

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Hydrodynamic theory

Variational principle

$$\log \mathbb{P}[\rho_0(x) \to \rho_T(x)] \sim \max_{\rho(x,t)} - \int_0^T dt \int dx \frac{(j + D(\rho)\partial_x \rho)^2}{2\sigma(\rho)}$$

Scaling limit of various diffusive systems :

$$D = 1/2, \, \sigma = \rho$$

$$D=1/2,\ \sigma=\rho(1-\rho)$$

$$D=1/2$$
, $\sigma=\rho^2$

⇒ notion of a general diffusive system

Hydrodynamic

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Known results

Open system

Bodineau Derrida 2004



$$\frac{1}{T}\log\left\langle e^{\lambda Q_{T}}\right\rangle \rightarrow -\frac{K}{L}\left[\int_{\rho_{a}}^{\rho_{b}}\frac{D(\rho)d\rho}{\sqrt{1+2K\sigma(\rho)}}\right]^{2}$$

where
$$K$$
 is given by $\lambda = \int_{\rho_a}^{\rho_b} \frac{d\rho D(\rho)}{2\sigma(\rho)} \left[\frac{1}{\sqrt{1+2K\sigma(\rho)}} - 1 \right]$

⇒ Non-universal distribution across models

Open system

Known results

Bodineau Derrida 2004

$$\rho_a$$

$$\frac{1}{T}\log\left\langle e^{\lambda Q_T}\right\rangle \to -\frac{K}{L}\left[\int_{\rho_s}^{\rho_b}\frac{D(\rho)d\rho}{\sqrt{1+2K\sigma(\rho)}}\right]^2$$

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- ⇒ Non-universal distribution across models
 - For the SSEP $(D = 1/2, \sigma = \rho(1 \rho))$:

$$\frac{1}{T}\log\left\langle e^{\lambda Q_T}\right\rangle_{\rho_{\mathrm{a}},\rho_{\mathrm{b}}} \rightarrow \left[\log\left(\sqrt{1+\omega}-\sqrt{\omega}\right)\right]^2$$

with
$$\omega = \rho_{\mathsf{a}}(e^{\lambda}-1) + \rho_{\mathsf{b}}(e^{-\lambda}-1) + \rho_{\mathsf{a}}\rho_{\mathsf{b}}(e^{\lambda}-1)(e^{-\lambda}-1)$$

Known results

Known results

Open system

Bodineau Derrida 2004



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where K is given by $\lambda = \int_{\rho_s}^{\rho_b} \frac{d\rho D(\rho)}{2\sigma(\rho)} \left| \frac{1}{\sqrt{1+2K\sigma(\rho)}} - 1 \right|$

- ⇒ Non-universal distribution across models
 - For the SSEP $(D = 1/2, \sigma = \rho(1 \rho))$:

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⇒ Coincides with the universal flux distribution for fermions in disordered conductor (Lee Yevitov Yakolev 95)

Discrete mod

Hydrodynami

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Known results

Ring

Appert D. Lecomte van Wijland 2008



$$\frac{1}{T}\log\left\langle e^{\lambda Q_T}\right\rangle - \frac{\lambda^2}{2}\frac{\left\langle Q_T^2\right\rangle}{T} \sim \frac{1}{L^2}\mathcal{F}\left(-\frac{\sigma\lambda^2}{4}\right)$$

where
$$\mathcal{F}(u) = -4\sum_{n\geq 1}\left[n\pi\sqrt{n^2\pi^2 - 2u} - n^2\pi^2 + u\right]$$

 $\Rightarrow \langle Q_T^n\rangle_c$ is universal for $n\geq 4$

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Known results

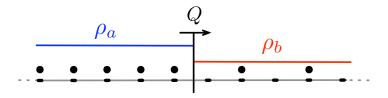
Discrete mo

Hydrodynamic

Results of this talk

Infinite line

Derrida Gerschenfeld 2009



For the SSEP only:

$$\log \left\langle e^{\lambda Q_t} \right\rangle \sim \sqrt{rac{t}{2\pi}} \sum_{n \geq 1} rac{(-)^{n+1}}{n^{3/2}} \omega^n$$

with
$$\omega = \rho_a(e^{\lambda} - 1) + \rho_b(e^{-\lambda} - 1) + \rho_a\rho_b(e^{\lambda} - 1)(e^{-\lambda} - 1)$$

The SSE

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Hydrodynamic theo

Known results

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The 551

Dynamic

Hydrodynamic theo

Discrete mo

Discrete mo

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with
$$\omega = \rho_a(e^{\lambda} - 1) + \rho_b(e^{-\lambda} - 1) + \rho_a\rho_b(e^{\lambda} - 1)(e^{-\lambda} - 1)$$

• The invariance $\langle e^{\lambda Q_t} \rangle_{\rho_a,\rho_b} = F_t(\omega)$ can be derived both from the discrete and continuous models

Known results

Results of this talk

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- The invariance $\langle e^{\lambda Q_t} \rangle_{\rho_1,\rho_h} = F_t(\omega)$ can be derived both from the discrete and continuous models
- The exact expression of $F_t(\omega)$, however, has only been derived in the discrete case

The 551

Dynamic

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Discrete mo

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Results of this talk

Infinite line

Derrida Gerschenfeld 2009

For the $\ensuremath{\mathsf{SSEP}}$ only :

$$\log \left\langle e^{\lambda Q_t} \right\rangle \sim \sqrt{\frac{t}{2\pi}} \sum_{n \geq 1} \frac{(-)^{n+1}}{n^{3/2}} \omega^n$$

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- The invariance $\langle e^{\lambda Q_t} \rangle_{\rho_a,\rho_b} = F_t(\omega)$ can be derived both from the discrete and continuous models
- The exact expression of $F_t(\omega)$, however, has only been derived in the discrete case
- But its asymptotics for large Q_t can be understood :

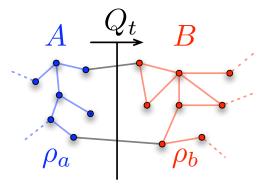
$$\mathbb{P}[Q_t \sim Q] symp = \exp\left[-rac{\pi^2}{12\sqrt{2}}rac{Q^3}{t}
ight]$$

Tail behavio

Conclusion

ω invariance for the discrete model

Flux Q_t between two regions during time t for any geometry :



Then
$$\left\langle e^{\lambda Q_t} \right\rangle_{\rho_a,\rho_b} = F_t(\omega)$$
 where $\omega = \rho_a(e^{\lambda} - 1) + \rho_b(e^{-\lambda} - 1) + \rho_a\rho_b(e^{\lambda} - 1)(e^{-\lambda} - 1)$

Calculation

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Conclusion

ω invariance for the discrete model

Derivation

ullet Two different expansions of $\left\langle e^{\lambda Q_t}
ight
angle$:

$$\langle e^{\lambda Q_t} \rangle = \sum_{n \geq 0} \frac{\lambda^n}{n!} \langle Q_t^n \rangle = \sum_{p,q \geq 0} \rho_a^p \rho_b^q g_{pq}(\lambda, t)$$

 ω invariance

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Conclusion

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• $\langle Q_t^n \rangle$ is a polynomial of degree n in ρ_a , ρ_b .

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Conclusion

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- $\langle Q_t^n \rangle$ is a polynomial of degree n in ρ_a , ρ_b .
- $g_{pq}(\lambda, t)$ is a linear combination of $(e^{-\lambda q}, ..., e^{\lambda p})$

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- $\langle Q_t^n \rangle$ is a polynomial of degree n in ρ_a , ρ_b .
- $g_{pq}(\lambda, t)$ is a linear combination of $(e^{-\lambda q}, ..., e^{\lambda p})$
- \Rightarrow For g_{pq} not to appear in $\langle Q_t^n \rangle$ for p+q>n, one needs

$$g_{pq}(\lambda, t) = (e^{\lambda} - 1)^{p}(e^{-\lambda} - 1)^{q}c_{pq}(t)$$

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Discrete m

ω invariance

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Conclusion

ω invariance for the discrete model

Derivation

• Two different expansions of $\left\langle e^{\lambda Q_t} \right\rangle$:

$$\left\langle e^{\lambda Q_t} \right\rangle = \sum_{n\geq 0} \frac{\lambda^n}{n!} \left\langle Q_t^n \right\rangle = \sum_{p,q\geq 0} \rho_a^p \rho_b^q \, g_{pq}(\lambda,t)$$

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$$g_{pq}(\lambda, t) = (e^{\lambda} - 1)^{p}(e^{-\lambda} - 1)^{q}c_{pq}(t)$$

• Now
$$\langle e^{\lambda Q_t} \rangle = G_t(\rho_a(e^{\lambda} - 1), \rho_b(e^{-\lambda} - 1)) = G_t(\alpha, \beta)$$

The SSEE

Discrete m

 ω invariance

Tail beh

Conclusion

ω invariance for the discrete model

•
$$\langle e^{\lambda Q_t} \rangle = G_t(\rho_a(e^{\lambda} - 1), \rho_b(e^{-\lambda} - 1)) = G_t(\alpha, \beta)$$

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Hydrodynam

Conclusion

ω invariance for the discrete model

- $\langle e^{\lambda Q_t} \rangle = G_t(\rho_a(e^{\lambda}-1), \rho_b(e^{-\lambda}-1)) = G_t(\alpha, \beta)$
- In the SSEP, the holes follow the same dynamics as the particles :

$$\left\langle e^{\lambda Q_t}\right\rangle_{\rho_a,\rho_b} = \left\langle e^{-\lambda Q_t}\right\rangle_{1-\rho_a,1-\rho_b}$$

The SSEF

Discrete m

ω invariance

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Conclusion

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$$\Rightarrow$$
 Here : $G_t(\alpha,\beta) = G_t(e^{-\lambda}(\alpha+1)-1,e^{\lambda}(\beta+1)-1)$ for any λ

The SSEF

Discrete m

ω invariance

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Conclusio

ω invariance for the discrete model

- $\left\langle e^{\lambda Q_t} \right\rangle = G_t(
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$$\Rightarrow G_t(\alpha,\beta) = G_t(\alpha\beta + \alpha + \beta,0) = G_t(\omega,0)$$
 and

ω invariance for the discrete model

- $\langle e^{\lambda Q_t} \rangle = G_t(\rho_a(e^{\lambda} 1), \rho_b(e^{-\lambda} 1)) = G_t(\alpha, \beta)$
- In the SSEP, the holes follow the same dynamics as the particles:

$$\left\langle e^{\lambda Q_t} \right\rangle_{\rho_a,\rho_b} = \left\langle e^{-\lambda Q_t} \right\rangle_{1-\rho_a,1-\rho_b}$$

$$\Rightarrow \text{ Here }: \ G_t(\alpha,\beta) = G_t(e^{-\lambda}(\alpha+1)-1,e^{\lambda}(\beta+1)-1) \text{ for any } \lambda$$

$$\Rightarrow G_t(\alpha,\beta) = G_t(\alpha\beta+\alpha+\beta,0) = G_t(\omega,0) \text{ and}$$

$$\left\langle e^{\lambda Q_t} \right\rangle_{\alpha,(\beta)} = F_t(\omega)$$

riyaroaynan

Conclusion

Calculation for the discrete model

• For $\rho_a=1, \rho_b=0$, the Bethe Ansatz (Tracy - Widom 2007-9) gives after some work

$$\left\langle e^{\lambda Q_t} \right\rangle = \det(I + \omega K_t)$$

with an operator

$$K_t f(z) = \oint_{|z'|=r \ll 1} \frac{dz'}{2i\pi} \frac{e^{\frac{t}{2}(z+1/z-2)}}{zz'+1-2z} f(z')$$

Calculation for the discrete model

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with an operator

$$K_t f(z) = \oint_{|z'| = r \ll 1} \frac{dz'}{2i\pi} \frac{e^{\frac{t}{2}(z+1/z-2)}}{zz' + 1 - 2z} f(z')$$

• When $t \to \infty$, ${\rm tr} {\cal K}^n_t \sim \sqrt{\frac{t}{2\pi n}}$ so that

$$\log\left\langle e^{\lambda Q_t}
ight
angle \sim \sqrt{rac{t}{2\pi}} \sum_{n\geq 1} rac{(-)^{n+1}}{n^{3/2}} \omega^n = \sqrt{rac{t}{2}} \int rac{dk}{\pi} \log(1+\omega e^{-k^2})$$

Tail behavior

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Conclusion

Tail of the distribution

• For large λ and Q_t , this gives

$$\log\left\langle e^{\lambda Q_t}
ight
angle \sim rac{2\sqrt{2t}}{\pi}\lambda^{3/2} \Rightarrow \mathbb{P}[Q_t \simeq Q] symp \exp\left[-rac{\pi^2}{12\sqrt{2}}rac{Q^3}{t}
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Hydrodynam

Conclusion

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• The scaling of fluctuations in $t^{1/3}$ had been observed in TASEP (Schutz, Johansson)

The SSEE

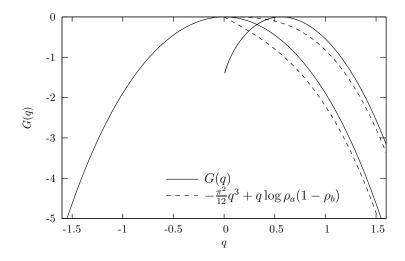
Discrete mod

Tail behavior

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Conclusion

Tail of the distribution



The SSEP

D:----

Organization and additional

Hydrodynamics Setting

Fluctuation theo

Conclusion

Hydrodynamic theory

Setting

$$\log \left\langle e^{\lambda Q_T} \right\rangle = \max_{\rho} - \iint dx dt \frac{(j + D(\rho)\partial_x \rho)^2}{2\sigma(\rho)} + \lambda \int_0^\infty dx [\rho(x, T) - \rho(x, 0)] - \int_{-\infty}^\infty dx S(\rho(x, 0))$$

Conclusio

Hydrodynamic theory

Setting

$$\log \left\langle e^{\lambda Q_T} \right\rangle = \max_{\rho} - \iint dx dt \frac{(j + D(\rho)\partial_x \rho)^2}{2\sigma(\rho)} + \lambda \int_0^\infty dx [\rho(x, T) - \rho(x, 0)] - \int_{-\infty}^\infty dx \, S(\rho(x, 0))$$

Counts the integrated current during time T

The SSEP

Discrete mode

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Setting

Fluctuation the

 ω -dependence

Conclusio

Hydrodynamic theory

Setting

$$\log \left\langle e^{\lambda Q_T} \right\rangle = \max_{\rho} - \iint dx dt \frac{(j + D(\rho)\partial_x \rho)^2}{2\sigma(\rho)} + \lambda \int_0^\infty dx [\rho(x, T) - \rho(x, 0)] - \int_{-\infty}^\infty dx \, S(\rho(x, 0))$$

- Counts the integrated current during time T
- Entropy of the initial state :

$$S(\rho) = f(\rho) - f(\bar{\rho}) - (\rho - \bar{\rho})f'(\bar{\rho})$$

$$\bar{\rho} = \begin{cases} \rho_a & \text{if } x < 0 \\ \rho_b & \text{if } x > 0 \end{cases} : \text{average initial density}$$

$$f$$
: entropy function $\left(f''(\rho) = \frac{2D(\rho)}{\sigma(\rho)}\right)$

The SSEE

Hydrodynami

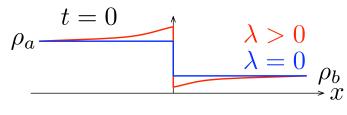
Setting

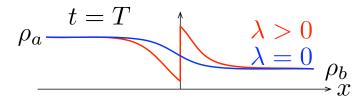
Fluctuation theo

Conclusio

Hydrodynamic theory

Setting





 ω -dependence

Conclusion

Hydrodynamic theory

Fluctuation theorem

• The cross-term in $(j + D\partial_x \rho)^2/2\sigma$ is a boundary term :

$$\iint dx dt \frac{jD\partial_x \rho}{\sigma} = \int dx \frac{\left[f(\rho)\right]_0^T}{2}$$

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Hydrodynami

Fluctuation theorem

 ω -dependence

Conclusion

Hydrodynamic theory

Fluctuation theorem

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$$\iint dxdt \frac{jD\partial_x \rho}{\sigma} = \int dx \frac{\left[f(\rho)\right]_0^T}{2}$$

• Since $[S(\rho)]_0^T = [f(\rho)]_0^T - f'(r)[\rho]_0^T$,

$$\log \left\langle e^{\lambda Q_T} \right\rangle = \max_{\rho} \int dx (\lambda \theta(x) - \frac{1}{2} f'(\bar{\rho}(x))) \left[\rho(x, t) \right]_0^T$$
$$- \frac{1}{2} \int dx (S(\rho(x, 0)) + S(\rho(x, T))) - \iint dx dt \frac{j^2 + (D\partial_x \rho)^2}{2\sigma}$$

Fluctuation theorem

 ω -dependence

Conclusion

Hydrodynamic theory

Eluctuation theorem

$$\log \left\langle e^{\lambda Q_T} \right\rangle = \max_{\rho} \int dx (\lambda \theta(x) - \frac{1}{2} f'(\bar{\rho}(x))) \left[\rho(x, t) \right]_0^T$$
$$- \frac{1}{2} \int dx (S(\rho(x, 0)) + S(\rho(x, T))) - \iint dx dt \frac{j^2 + (D\partial_x \rho)^2}{2\sigma}$$

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Setting Fluctuation theorem

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Conclusion

Hydrodynamic theory

Fluctuation theorem

$$\log \left\langle e^{\lambda Q_T} \right\rangle = \max_{\rho} \int dx (\lambda \theta(x) - \frac{1}{2} f'(\bar{\rho}(x))) \left[\rho(x, t) \right]_0^T$$
$$- \frac{1}{2} \int dx (S(\rho(x, 0)) + S(\rho(x, T))) - \iint dx dt \frac{j^2 + (D\partial_x \rho)^2}{2\sigma}$$

• Reverse time :
$$\left\{ egin{array}{l}
ho(x,t)
ightarrow
ho(x,T-t) \ j(x,t)
ightarrow -j(x,T-t) \end{array}
ight.$$

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Conclusion

Hydrodynamic theory

Fluctuation theorem

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 ho(x,T-t) \ j(x,t)
 ightarrow -j(x,T-t) \end{array}
 ight.$
- One term changes \Rightarrow new λ :

$$\left\langle e^{\lambda Q_T} \right\rangle = \left\langle e^{\lambda' Q_T} \right\rangle \text{ for } \lambda + \lambda' = \int_{\rho_s}^{\rho_b} d\rho \frac{2D(\rho)}{\sigma(\rho)}$$

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Setting

Fluctuation theorem

 ω -dependence

Conclusion

Hydrodynamic theory

Fluctuation theorem

$$\log \left\langle e^{\lambda Q_T} \right\rangle = \max_{\rho} \int dx (\lambda \theta(x) - \frac{1}{2} f'(\bar{\rho}(x))) \left[\rho(x, t) \right]_{0}^{T}$$
$$- \frac{1}{2} \int dx (S(\rho(x, 0)) + S(\rho(x, T))) - \iint dx dt \frac{j^2 + (D\partial_x \rho)^2}{2\sigma}$$

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- One term changes \Rightarrow new λ :

$$\left\langle e^{\lambda Q_T} \right\rangle = \left\langle e^{\lambda' Q_T} \right\rangle \text{ for } \lambda + \lambda' = \int_{\rho_a}^{\rho_b} d\rho \frac{2D(\rho)}{\sigma(\rho)}$$

⇒ A fluctuation theorem (Galavotti Cohen, ...) out of stationnary state

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$$\log \left\langle e^{\lambda Q_T} \right\rangle = \max_{\rho} \int dx (\lambda \theta(x) - \frac{1}{2} f'(\bar{\rho}(x))) \left[\rho(x, t) \right]_{0}^{T}$$
$$- \frac{1}{2} \int dx (S(\rho(x, 0)) + S(\rho(x, T))) - \iint dx dt \frac{j^2 + (D\partial_x \rho)^2}{2\sigma}$$

- Reverse time : $\left\{ egin{array}{l}
 ho(x,t)
 ightarrow
 ho(x,T-t) \ j(x,t)
 ightarrow -j(x,T-t) \end{array}
 ight.$
- One term changes \Rightarrow new λ :

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ho_a}^{
ho_b} d
ho rac{2D(
ho)}{\sigma(
ho)}$$

- \Rightarrow A fluctuation theorem (Galavotti Cohen, ...) out of stationnary state
 - For the SSEP : $\lambda' = -\lambda + \log \frac{\rho_a (1-\rho_b)}{(1-\rho_a)\rho_b}$ gives the same value of ω

 ω -dependence

Conclusio

Hydrodynamics theory

ω -dependence for the SSEP

• Introduce an auxiliary field $\hat{\rho}$:

$$\max_{\rho} \iint -\frac{(j+D\partial_{x}\rho)^{2}}{2\sigma} = \max_{\rho,\hat{\rho}} \iint \frac{(j+D\partial_{x}\rho+\sigma\partial_{x}\hat{\rho})^{2}-(j+D\partial_{x}\rho)^{2}}{2\sigma}$$
$$= \max_{\rho,\hat{\rho}} \iint \hat{\rho}\partial_{t}\rho+D\partial_{x}\rho\partial_{x}\hat{\rho}+\frac{\sigma}{2}(\partial_{x}\hat{\rho})^{2}$$

Conclusio

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• Boundary equations on $(\rho, \hat{\rho})$ at t = 0, T:

$$\begin{cases} \hat{\rho}(x,T) &= -\lambda \theta(x) \\ \hat{\rho}(x,0) &= -\lambda \theta(x) - f'(\bar{\rho}(x)) - f'(\rho(x,0)) + C \end{cases}$$

ω-dependence

Hydrodynamics and ω

ω -dependence for the SSEP

• For the SSEP : $D = 1/2, \sigma = \rho(1-\rho)$. Take

$$\left\{ \begin{array}{ll} S_{+} &= \rho e^{\hat{\rho}} \\ S_{-} &= (1-\rho)e^{-\hat{\rho}} \\ S_{z} &= \rho - 1/2 \end{array} \right. \text{ and } \vec{S}.\vec{S}' = \frac{1}{2} \left(S_{+}S'_{-} + S_{-}S'_{+} \right) + S_{z}S'_{z}$$

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⇒ Then the bulk integrand becomes

$$\hat{\rho}\partial_t \rho + D\partial_x \rho \partial_x \hat{\rho} + \frac{\sigma}{2} (\partial_x \hat{\rho})^2 = \hat{\rho}\partial_t \rho - \frac{1}{2} \frac{\partial_x \vec{S}}{\partial_x \vec{S}} \frac{\vec{S}}{\partial_x \vec{S}}$$

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 \Rightarrow Orthogonal transforms on \vec{S} give families of optima for the bulk integral

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ω -dependence for the SSEP : Boundary conditions

Objective : Find A orthogonal such that

$$\begin{pmatrix} (\rho_{\mathsf{a}}, \rho_{\mathsf{b}}, \lambda) \\ \rho, \hat{\rho}, \vec{\mathsf{S}} \end{pmatrix} \xrightarrow{\mathsf{A}} \begin{cases} (\rho_{\mathsf{a}}', \rho_{\mathsf{b}}', \lambda') \\ \rho', \hat{\rho}', \vec{\mathsf{S}}' \end{cases}$$

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Necessary condition: from the boundary equations.

$$ec{S}(-\infty,t) = \left(egin{array}{c}
ho_a \ 1-
ho_a \
ho_a-1/2 \end{array}
ight) \ \ {
m and} \ \ ec{S}(+\infty,t) = \left(egin{array}{c}
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ω -dependence for the SSEP : Boundary conditions

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 \Rightarrow The transform only works at constant ω

Conclusion

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ω -dependence for the SSEP : Boundary conditions

• Express ρ , $\hat{\rho}$ for $(\rho_a, \rho_b, \lambda)$ from $(\mu, \hat{\mu})$ for $\rho_a = \rho_b = 1/2$ at the same ω :

$$\rho = \frac{\left(e^{-\hat{\mu}} \operatorname{sh} \frac{\lambda + u - v}{2} - \operatorname{sh} \frac{\lambda - u - v}{2}\right) \left(\mu e^{\hat{\mu}} \operatorname{sh} \frac{u + v}{2} - (1 - \mu) \operatorname{sh} \frac{u - v}{2}\right)}{\operatorname{sh} u \operatorname{sh} \frac{\lambda}{2}}$$

$$e^{\hat{\mu}} = \frac{e^{\hat{\mu}} (1 - e^{u + v}) - e^{u} + e^{v}}{e^{\hat{\mu}} (e^{\lambda/2} - e^{u + v - \lambda/2}) + e^{v - \lambda/2} - e^{u + \lambda/2}}$$

where
$$\rho_a = (e^{\nu} \operatorname{ch} u - 1)/(e^{\lambda} - 1), \ \rho_b = (e^{-\nu} \operatorname{ch} u - 1)/(e^{-\lambda} - 1)$$

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ω -dependence for the SSEP : Boundary conditions

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$$\rho = \frac{\left(e^{-\hat{\mu}} \sinh \frac{\lambda + u - v}{2} - \sinh \frac{\lambda - u - v}{2}\right) \left(\mu e^{\hat{\mu}} \sinh \frac{u + v}{2} - (1 - \mu) \sinh \frac{u - v}{2}\right)}{\sinh u \sinh \frac{\lambda}{2}}$$

$$e^{\hat{\mu}} = \frac{e^{\hat{\mu}} (1 - e^{u + v}) - e^{u} + e^{v}}{e^{\hat{\mu}} (e^{\lambda/2} - e^{u + v - \lambda/2}) + e^{v - \lambda/2} - e^{u + \lambda/2}}$$

where $\rho_a = (e^v \operatorname{ch} u - 1)/(e^{\lambda} - 1)$, $\rho_b = (e^{-v} \operatorname{ch} u - 1)/(e^{-\lambda} - 1)$ \Rightarrow Only need to calculate for $\rho_a = \rho_b = 1/2$

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Discrete models

- SSEP : ω invariance, resolution of the Bethe Ansatz
- Other models: invariances, explicit expressions..?

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Hydrodynamic theory

- SSEP: all initial conditions are related
- But one still needs to solve at least one...

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Discrete models

- SSEP : ω invariance, resolution of the Bethe Ansatz
- Other models: invariances, explicit expressions..?

Hydrodynamic theory

- SSEP: all initial conditions are related
- But one still needs to solve at least one...
- General diffusive system : a fluctuation theorem, but no invariances or large deviation functionnals
- Possibly generalize the distribution's decay :

$$\mathbb{P}[Q_t \simeq Q] symp = \exp\left[-lpha rac{Q^3}{t}
ight]$$