Invariances and large deviations of the current for the SSEP on $\mathbb{Z}$

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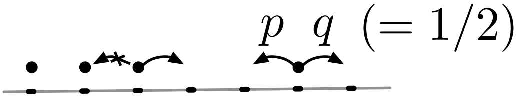
The simple exclusion process

Microscopic dynamics



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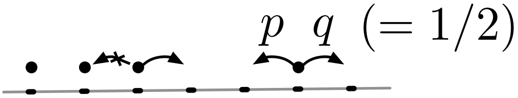
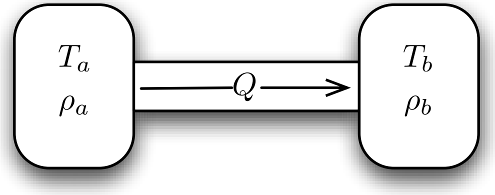


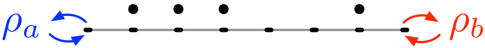
Simple transport model



$\Rightarrow$ Distribution of Q ?

Geometries

Open system



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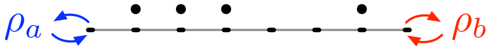
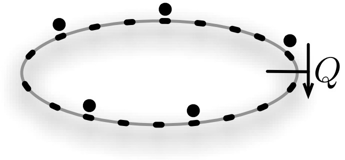


Fluctuations on a ring



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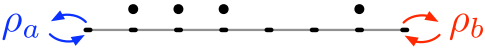

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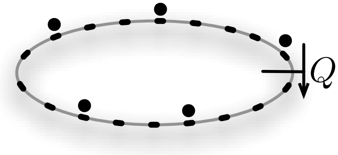
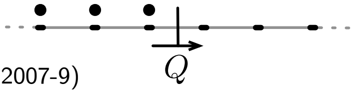


Step initial condition on $\mathbb{Z}$



SSEP : Spohn (1989), Tracy Widom (2007-9)

TASEP : Schutz (1993), Johansson (2000), Prahofer Spohn (2000)

Hydrodynamic theory

Variational principle

Kipnis Olla Varadhan 89, Spohn 91, Bertini DeSole Gabrielli Jona-Lasinio Landim 01

- Continuous model for diffusive systems
- Conserved particle density $\rho(x,t)$
- $\rho(x,t)$ diffuses on average, with excursions :

$$\log \mathbb{P}[\rho_0(x) \to \rho_T(x)] \sim \max_{\rho(x,t)} -\int_0^T dt \int dx \frac{(j + D(\rho)\partial_x \rho)^2}{2\sigma(\rho)}$$

- $j(x,t)$: current
- Conservation : $\partial_x j + \partial_t \rho = 0$
- On average, $j = -D(\rho)\partial_x \rho$ (Fick's law)

Hydrodynamic theory

Variational principle

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Scaling limit of various diffusive systems :

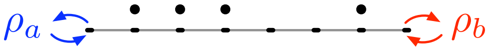
- Non-interacting particles : $D = 1/2, \ \sigma = \rho$
- Symmetric exclusion : $D = 1/2, \ \sigma = \rho(1-\rho)$
- Kipnis-Marchioro-Presutti model : $D = 1/2, \ \sigma = \rho^2$

$\Rightarrow$ notion of a general diffusive system

Known results

Open system

Bodineau Derrida 2004



$$\frac{1}{T}\log\left\langle e^{\lambda Q_T}\right\rangle \to -\frac{K}{L}\left[\int_{\rho_a}^{\rho_b}\frac{D(\rho)d\rho}{\sqrt{1+2K\sigma(\rho)}}\right]^2$$

where K is given by $\lambda = \int_{\rho_a}^{\rho_b}\frac{d\rho D(\rho)}{2\sigma(\rho)}\left[\frac{1}{\sqrt{1+2K\sigma(\rho)}}-1\right]$

$\Rightarrow$ Non-universal distribution across models

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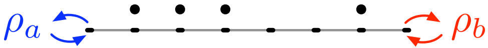

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$\Rightarrow$ Non-universal distribution across models

- For the SSEP ($D = 1/2$, $\sigma = \rho(1-\rho)$) :

$$\frac{1}{T}\log\left\langle e^{\lambda Q_T}\right\rangle_{\rho_a,\rho_b} \to \left[\log\left(\sqrt{1+\omega}-\sqrt{\omega}\right)\right]^2$$

with $\omega = \rho_a(e^{\lambda}-1)+\rho_b(e^{-\lambda}-1)+\rho_a\rho_b(e^{\lambda}-1)(e^{-\lambda}-1)$

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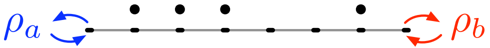

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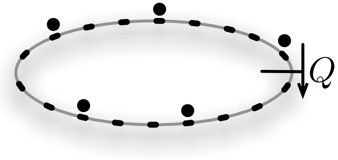
$\Rightarrow$ Coincides with the universal flux distribution for fermions in disordered conductor (Lee Yevitov Yakolev 95)

Known results

Ring

Appert D. Lecomte van Wijland 2008



$$\frac{1}{T}\log\left\langle e^{\lambda Q_T}\right\rangle - \frac{\lambda^2}{2}\frac{\langle Q_T^2\rangle}{T} \sim \frac{1}{L^2}\mathcal{F}\left(-\frac{\sigma\lambda^2}{4}\right)$$

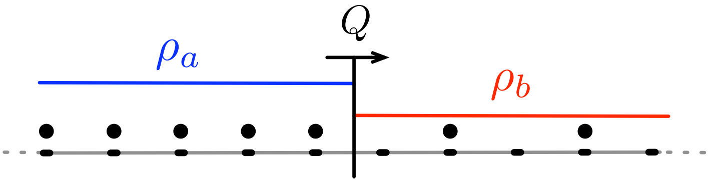
where $\mathcal{F}(u) = -4\sum_{n\geq 1}\left[n\pi\sqrt{n^2\pi^2 - 2u} - n^2\pi^2 + u\right]$

$\Rightarrow \langle Q_T^n\rangle_c$ is universal for $n \geq 4$

Results of this talk

Infinite line

Derrida Gerschenfeld 2009



For the SSEP only :

$$\log \left\langle e^{\lambda Q_t} \right\rangle \sim \sqrt{\frac{t}{2\pi}} \sum_{n \geq 1} \frac{(-)^{n+1}}{n^{3/2}} \omega^n$$

with $\omega = \rho_a(e^{\lambda} - 1) + \rho_b(e^{-\lambda} - 1) + \rho_a \rho_b(e^{\lambda} - 1)(e^{-\lambda} - 1)$

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- The invariance $\left\langle e^{\lambda Q_t} \right\rangle_{\rho_a,\rho_b} = F_t(\omega)$ can be derived both from the discrete and continuous models

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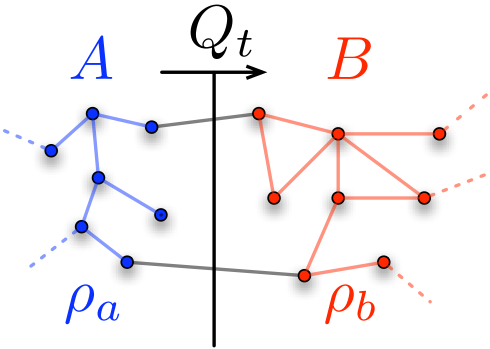
- The invariance $\left\langle e^{\lambda Q_t} \right\rangle_{\rho_a, \rho_b} = F_t(\omega)$ can be derived both from the discrete and continuous models
- The exact expression of $F_t(\omega)$, however, has only been derived in the discrete case
- But its asymptotics for large Q_t can be understood :

$$\mathbb{P}[Q_t \sim Q] \asymp \exp\left[-\frac{\pi^2}{12\sqrt{2}} \frac{Q^3}{t}\right]$$

ω invariance for the discrete model

Flux Q_t between two regions during time t for any geometry :



Then $\left\langle e^{\lambda Q_t} \right\rangle_{\rho_a,\rho_b} = F_t(\omega)$ where

$$\omega = \rho_a(e^{\lambda} - 1) + \rho_b(e^{-\lambda} - 1) + \rho_a\rho_b(e^{\lambda} - 1)(e^{-\lambda} - 1)$$

ω invariance for the discrete model

Derivation

- Two different expansions of $\left\langle e^{\lambda Q_t} \right\rangle$:

$$\left\langle e^{\lambda Q_t} \right\rangle = \sum_{n \geq 0} \frac{\lambda^n}{n!} \left\langle Q_t^n \right\rangle = \sum_{p,q \geq 0} \rho_a^p \rho_b^q \, g_{pq}(\lambda, t)$$

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- $g_{pq}(\lambda, t)$ is a linear combination of $(e^{-\lambda q}, .., e^{\lambda p})$

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$\Rightarrow$ For g_{pq} not to appear in $\langle Q_t^n \rangle$ for $p + q > n$, one needs

$$g_{pq}(\lambda, t) = (e^{\lambda} - 1)^p (e^{-\lambda} - 1)^q c_{pq}(t)$$

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- Now $\left\langle e^{\lambda Q_t}\right\rangle = G_t(\rho_a(e^{\lambda} - 1), \rho_b(e^{-\lambda} - 1)) = G_t(\alpha, \beta)$

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- $\left\langle e^{\lambda Q_t} \right\rangle = G_t(\rho_a(e^{\lambda} - 1), \rho_b(e^{-\lambda} - 1)) = G_t(\alpha, \beta)$
- In the SSEP, the holes follow the same dynamics as the particles :

$$\left\langle e^{\lambda Q_t} \right\rangle_{\rho_a,\rho_b} = \left\langle e^{-\lambda Q_t} \right\rangle_{1-\rho_a,1-\rho_b}$$

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$$\left\langle e^{\lambda Q_t} \right\rangle_{\rho_a,\rho_b} = F_t(\omega)$$

Calculation for the discrete model

- For $\rho_a = 1, \rho_b = 0$, the Bethe Ansatz (Tracy - Widom 2007-9) gives after some work

$$\left\langle e^{\lambda Q_t} \right\rangle = \det(I + \omega K_t)$$

with an operator

$$K_t f(z) = \oint_{|z'|=r\ll 1} \frac{dz'}{2i\pi} \frac{e^{\frac{t}{2}(z+1/z-2)}}{zz' + 1 - 2z} f(z')$$

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- When $t \to \infty$, $\mathrm{tr} K_t^n \sim \sqrt{\frac{t}{2\pi n}}$ so that

$$\log \left\langle e^{\lambda Q_t} \right\rangle \sim \sqrt{\frac{t}{2\pi}} \sum_{n \geq 1} \frac{(-)^{n+1}}{n^{3/2}} \omega^n = \sqrt{\frac{t}{2}} \int \frac{dk}{\pi} \log(1 + \omega e^{-k^2})$$

Tail of the distribution

- For large λ and Q_t, this gives

$$\log \left\langle e^{\lambda Q_t} \right\rangle \sim \frac{2\sqrt{2t}}{\pi}\lambda^{3/2} \Rightarrow \mathbb{P}[Q_t \simeq Q] \asymp \exp\left[-\frac{\pi^2}{12\sqrt{2}}\frac{Q^3}{t}\right]$$

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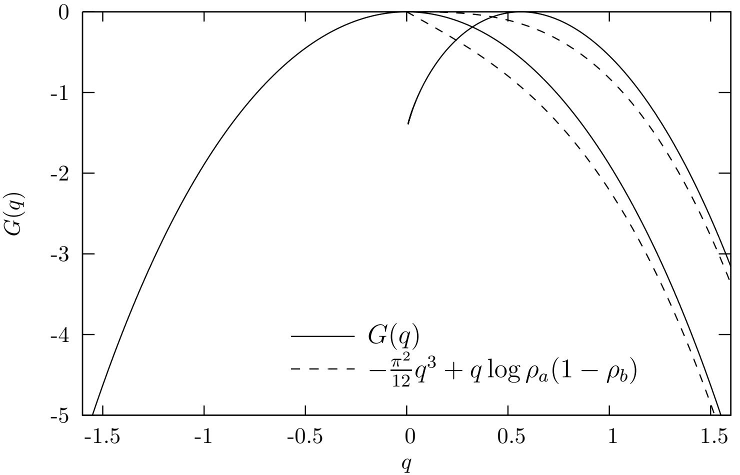
- The scaling of fluctuations in $t^{1/3}$ had been observed in TASEP (Schutz, Johansson)

Tail of the distribution



Hydrodynamic theory

Setting

$$\log \left\langle e^{\lambda Q_T} \right\rangle = \max_{\rho} - \iint dxdt \frac{(j + D(\rho)\partial_x \rho)^2}{2\sigma(\rho)} + \lambda \int_0^{\infty} dx [\rho(x,T) - \rho(x,0)] - \int_{-\infty}^{\infty} dx\, S(\rho(x,0))$$

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- Counts the integrated current during time T

Hydrodynamic theory

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- Counts the integrated current during time T
- Entropy of the initial state :

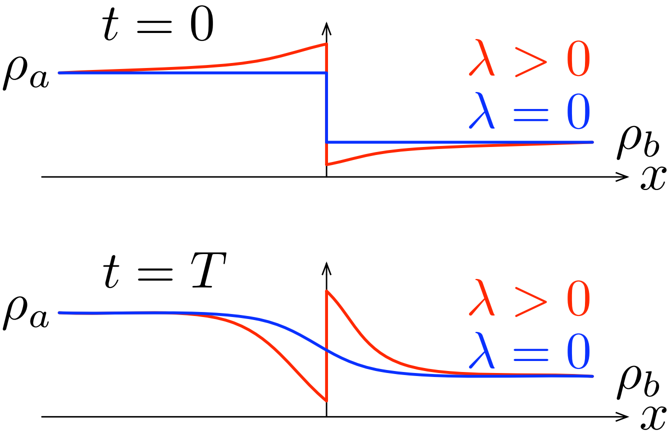
$$S(\rho) = f(\rho) - f(\bar{\rho}) - (\rho - \bar{\rho}) f'(\bar{\rho})$$

$$\bar{\rho} = \begin{cases} \rho_a & \text{if } x < 0 \\ \rho_b & \text{if } x > 0 \end{cases}$$: average initial density

f : entropy function $\left(f''(\rho) = \frac{2D(\rho)}{\sigma(\rho)} \right)$

Hydrodynamic theory

Setting



Hydrodynamic theory

Fluctuation theorem

- The cross-term in $(j + D\partial_x \rho)^2/2\sigma$ is a boundary term :

$$\iint dxdt \frac{jD\partial_x \rho}{\sigma} = \int dx \frac{\left[f(\rho)\right]_0^T}{2}$$

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- Since $\left[S(\rho)\right]_0^T = \left[f(\rho)\right]_0^T - f'(r)\left[\rho\right]_0^T$,

$$\log \left\langle e^{\lambda Q_T} \right\rangle = \max_{\rho} \int dx \left(\lambda\theta(x) - \frac{1}{2} f'(\bar{\rho}(x))\right)\left[\rho(x,t)\right]_0^T$$
$$- \frac{1}{2}\int dx \left(S(\rho(x,0)) + S(\rho(x,T))\right) - \iint dxdt \frac{j^2 + (D\partial_x\rho)^2}{2\sigma}$$

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- Reverse time : $\begin{cases} \rho(x,t) \to \rho(x,T-t) \\ j(x,t) \to -j(x,T-t) \end{cases}$

Hydrodynamic theory

Fluctuation theorem

$$\log \left\langle e^{\lambda Q_T} \right\rangle = \max_{\rho} \int dx(\lambda\theta(x) - \frac{1}{2} f'(\bar{\rho}(x))) \left[\rho(x,t)\right]_0^T$$
$$- \frac{1}{2} \int dx(S(\rho(x,0)) + S(\rho(x,T))) - \iint dxdt \frac{j^2 + (D\partial_x \rho)^2}{2\sigma}$$

- Reverse time : $\begin{cases} \rho(x,t) \to \rho(x,T-t) \\ j(x,t) \to -j(x,T-t) \end{cases}$
- One term changes $\Rightarrow$ new λ :

$$\left\langle e^{\lambda Q_T} \right\rangle = \left\langle e^{\lambda' Q_T} \right\rangle \text{ for } \lambda + \lambda' = \int_{\rho_a}^{\rho_b} d\rho \frac{2D(\rho)}{\sigma(\rho)}$$

Hydrodynamic theory

Fluctuation theorem

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$\Rightarrow$ A fluctuation theorem (Galavotti Cohen, ...) out of stationnary state

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$\Rightarrow$ A fluctuation theorem (Galavotti Cohen, ...) out of stationnary state

- For the SSEP : $\lambda' = -\lambda + \log \frac{\rho_a(1-\rho_b)}{(1-\rho_a)\rho_b}$ gives the same value of ω

Hydrodynamics theory

ω-dependence for the SSEP

- Introduce an auxiliary field $\hat{\rho}$:

$$\max_{\rho} \iint -\frac{(j + D\partial_x\rho)^2}{2\sigma} = \max_{\rho,\hat{\rho}} \iint \frac{(j + D\partial_x\rho + \sigma\partial_x\hat{\rho})^2 - (j + D\partial_x\rho)^2}{2\sigma}$$

$$= \max_{\rho,\hat{\rho}} \iint \hat{\rho}\partial_t\rho + D\partial_x\rho\partial_x\hat{\rho} + \frac{\sigma}{2}(\partial_x\hat{\rho})^2$$

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- Boundary equations on $(\rho, \hat\rho)$ at $t = 0, T$:

$$\begin{cases} \hat\rho(x,T) &= -\lambda\theta(x) & +C \\ \hat\rho(x,0) &= -\lambda\theta(x) - f'(\bar\rho(x)) - f'(\rho(x,0)) & +C \end{cases}$$

Hydrodynamics and ω

ω-dependence for the SSEP

- For the SSEP : $D = 1/2, \sigma = \rho(1-\rho)$. Take

$$\begin{cases} S_+ &= \rho e^{\hat\rho} \\ S_- &= (1-\rho)e^{-\hat\rho} \\ S_z &= \rho - 1/2 \end{cases} \quad \text{and } \vec{S}.\vec{S}' = \frac{1}{2}(S_+S'_- + S_-S'_+) + S_zS'_z$$

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$\Rightarrow$ Orthogonal transforms on $\vec{S}$ give families of optima for the bulk integral

Hydrodynamics and ω

ω-dependence for the SSEP : Boundary conditions

- **Objective** : Find A **orthogonal** such that

$$\left.\begin{array}{r}(\rho_a, \rho_b, \lambda) \\ \rho, \hat{\rho}, \vec{S}\end{array}\right\} \xrightarrow{A} \left\{\begin{array}{l}(\rho'_a, \rho'_b, \lambda') \\ \rho', \hat{\rho}', \vec{S}'\end{array}\right.$$

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$$\vec{S}(-\infty, t) = \begin{pmatrix} \rho_a \\ 1-\rho_a \\ \rho_a - 1/2 \end{pmatrix} \text{ and } \vec{S}(+\infty, t) = \begin{pmatrix} \rho_b e^{-\lambda} \\ (1-\rho_b)e^{\lambda} \\ \rho_b - 1/2 \end{pmatrix}$$

Hydrodynamics and ω

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$\Rightarrow$ The transform only works at constant ω

Hydrodynamics and ω

ω-dependence for the SSEP : Boundary conditions

- Express $\rho, \hat{\rho}$ for $(\rho_a, \rho_b, \lambda)$ from $(\mu, \hat{\mu})$ for $\rho_a = \rho_b = 1/2$ at the same ω :

$$\rho = \frac{\left(e^{-\hat{\mu}} \operatorname{sh} \frac{\lambda+u-v}{2} - \operatorname{sh} \frac{\lambda-u-v}{2}\right)\left(\mu e^{\hat{\mu}} \operatorname{sh} \frac{u+v}{2} - (1-\mu) \operatorname{sh} \frac{u-v}{2}\right)}{\operatorname{sh} u \operatorname{sh} \frac{\lambda}{2}}$$

$$e^{\hat{\rho}} = \frac{e^{\hat{\mu}}(1-e^{u+v}) - e^{u} + e^{v}}{e^{\hat{\mu}}(e^{\lambda/2} - e^{u+v-\lambda/2}) + e^{v-\lambda/2} - e^{u+\lambda/2}}$$

where $\rho_a = (e^{v} \operatorname{ch} u - 1)/(e^{\lambda} - 1)$, $\rho_b = (e^{-v} \operatorname{ch} u - 1)/(e^{-\lambda} - 1)$

Hydrodynamics and ω

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where $\rho_a = (e^{v} \operatorname{ch} u - 1)/(e^{\lambda} - 1)$, $\rho_b = (e^{-v} \operatorname{ch} u - 1)/(e^{-\lambda} - 1)$

$\Rightarrow$ Only need to calculate for $\rho_a = \rho_b = 1/2$

Conclusion

Discrete models

- SSEP : ω invariance, resolution of the Bethe Ansatz
- Other models : invariances, explicit expressions.. ?

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- But one still needs to solve at least one...

Conclusion

Discrete models

- SSEP : ω invariance, resolution of the Bethe Ansatz
- Other models : invariances, explicit expressions.. ?

Hydrodynamic theory

- SSEP : all initial conditions are related
- But one still needs to solve at least one...
- General diffusive system : a fluctuation theorem, but no invariances or large deviation functionnals
- Possibly generalize the distribution's decay :

$$\mathbb{P}[Q_t \simeq Q] \asymp \exp\left[-\alpha \frac{Q^3}{t}\right]$$